A fractional reaction-diffusion description of supply and demand

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Abstract. We suggest that the broad distribution of time scales in financial markets could be a crucial ingredient to reproduce realistic price dynamics in stylised Agent-Based Models. We propose a fractional reaction-diffusion model for the dynamics of liquidity in financial markets, where agents are very heterogeneous in terms of their characteristic frequencies. Several features of our model are amenable to an exact analytical treatment. We find in particular that the impact is a concave function of the transacted volume (*aka* the "square-root impact law"), as in the normal diffusion limit. However, the impact kernel decays as $t^{-\beta}$ with $\beta = 1/2$ in the diffusive case, which is inconsistent with market efficiency. In the sub-diffusive case the decay exponent β takes any value in $[0, 1/2]$, and can be tuned to match the empirical value $\beta \approx 1/4$. Numerical simulations confirm our theoretical results. Several extensions of the model are suggested.

1 Introduction

In the past few years, the concave nature of the impact of volume on prices – coined the "square-root impact law" – has made its way among the most firmly established stylized facts of modern finance [1,2,3,4,5,6].

Several attempts have been made to build theoretical models that account for non-linear market impact, see *e.g.* [7,8]. Following the ideas of Tóth *et al.* [3], the notion of a locally linear *latent* order-book model (LLOB) was introduced in [9,10]. The latter model builds upon coupled continuous reaction-diffusion equations for the dynamics of the bid and the ask sides of the latent order book and allows one to compute the price trajectory conditioned to any execution profiles. In the slow execution limit, the LLOB model was shown to match the linear propagator model that relates past order flow to price changes through a power law decaying kernel.

The propagator model was initially introduced in [11] to solve the so-called diffusivity puzzle: prices are approximately diffusive when the order flow is highly persistent [11,12,13]. In particular, the sign of the order-flow is characterised by an autocorrelation function $C(t)$ that decays as a power law $t^{-\gamma}$ with an exponent $\gamma < 1$, defining a *long-memory process*. Typically γ is found to be ≈ 0.5 for individual stocks. Naively, the impact of these correlated orders should lead to a super-diffusive price dynamics, with a Hurst exponent $H = 1 - \gamma/2 > 1/2$. In order to compensate for order flow correlation and restore price diffusivity, the impact of each order, described by a certain kernel or "propagator", must itself decay as a power law of time, with an exponent $\beta = (1 - \gamma)/2$ [11,13].

One major issue of the original LLOB framework is that its kernel decays with exponent $\beta = 1/2$, which cannot fulfil the above relation since $\gamma > 0$. Impact relaxation is too quick, and the price dynamics generated by the LLOB model exhibits significant *mean reversion* on short to medium time scales, a feature that is not observed in empirical data. However, the current version of the LLOB model postulates that market participants are homogeneous, in the sense that the distribution of volumes, reaction times, pricing updates, etc. are all *thin tailed*. For example, the cancellation of orders is assumed to be a Poisson process with a single cancellation rate ν.

In reality, different actors in financial markets are known to be highly heterogeneous with very widely distributed volumes and time scales, from High Frequency Traders (HFT) to large institutional investors. This observation suggests various generalisations of the LLOB. The path that we follow in this paper is that of a broad distribution of time scales for agents' intentions and re-evaluation, that naturally leads to a *fractional* reaction-diffusion process. Other possible generalisations are discussed in the conclusions.

The outline of the paper is as follows. We first present the general fractional diffusion framework with death and sources. We then present the fractional latent order-book model (FLOB) and derive its equilibrium shape, in the case of a balanced flow of buy/sell orders. We find that the characteristic V-shape of the latent order book is preserved, leading to a concave impact law. We also show that the corresponding propagator is now described by a tunable exponent $\beta \in [0, 1/2]$ that resolves the above mentioned diffusivity puzzle. We finally confront these results to numerical simulations.

2 Fractional diffusion with death and sources

The (fractional) diffusion equation describes random walkers that pause for a certain waiting time before resuming their motion. When the average waiting time is finite, the long-time, large scale description of an ensemble of such walkers is the standard diffusion equation. When the average waiting time diverges, the corresponding dynamics is described by the *fractional diffusion equation* [14,15]:

$$\partial_t \phi = K_\alpha \mathfrak{D}_t^{1-\alpha} \left(\partial_{xx}\phi\right) \ , \tag{1}$$

where K_α is a generalised diffusion coefficient, $\alpha < 1$ is the tail exponent of the waiting time distribution, $\mathfrak{D}_t^{-\alpha}$ denotes the fractional Riemann-Liouville operator [16,15] defined as $\mathfrak{D}_t^{-\alpha} f(t) = \Gamma(\alpha)^{-1} \int_0^t \mathrm{d}u\, (t-u)^{\alpha-1} f(u)$, and $\mathfrak{D}_t^{1-\alpha} = \partial_t \mathfrak{D}_t^{-\alpha}$. When $\alpha > 1$, the standard diffusion equation is recovered. Hence, in all the following equations involving α, one can formally read $\alpha \to \min(1,\alpha)$ where 1 corresponds to the standard diffusion case.

Let us now assume that, after waiting a power-law distributed time τ, a random walker can either resume its motion, or disappear, with some rate ν. The fractional diffusion equation then reads:

$$\partial_t \phi = K_\alpha \mathfrak{D}_t^{1-\alpha} \left(\partial_{xx}\phi - \nu\phi\right) + s(x,t) \ , \tag{2}$$

where $s(x,t)$ denotes is a general rate-source term allowing for the injection or removal of particles [17].

Taking the Fourier-Laplace transform ($\mathcal{FL}$) of Eq. (2) and letting $K_\alpha = K\Gamma(\alpha)$ yields:

$$\phi(k,p) = \frac{p^{\alpha-1}}{p^\alpha + K(k^2+\nu)} \big[\phi(k,0) + s(k,p)\big] \, . \tag{3}$$

The general solution in real space can thus be written as the sum of a convolution death/diffusion term and a source contribution:

$$\phi(x,t) = [\mathcal{G}_\alpha * \phi_0](x,t) + (\mathcal{FL})^{-1}\left\{\frac{p^{\alpha-1}s(k,p)}{p^\alpha + K(k^2+\nu)}\right\}, \tag{4}$$

where $\phi_0(x) = \phi(x, t=0)$ and where $\mathcal{G}_\alpha(x,t)$ denotes the inverse Fourier transform of the Mittag-Leffler[1] function [15,16], $\mathcal{G}_\alpha(x,t) = \mathcal{F}^{-1}\{F_\alpha[K(k^2+\nu)t^\alpha]\}$. Note that in the limit $\nu \to 0$ the kernel $\mathcal{G}_\alpha(x,t)$ can be conveniently written as:

$$\mathcal{G}_\alpha(x,t) = \frac{1}{\sqrt{4\pi K t^\alpha}}\, g_\alpha\left(\frac{x^2}{4Kt^\alpha}\right), \tag{5}$$

where we introduced the function g_α, the shape of which is discussed in *e.g.* [16,14].

[1] The Mittag-Leffler function $E_\alpha(z) = F_\alpha(-z)$ is a special function defined by the following series as: $F_\alpha(z) = \sum_{j=0}^{\infty} \frac{(-z)^j}{\Gamma(1+j\alpha)}$.

3 Fractional latent order-book model

Following the assumptions of Donier *et al.* [10,18], we posit that the dynamics of the intentions of market participants results from order cancellation or reassessment of their reservation price. The crucial difference is that we now assume that such events occur after a fat-tailed waiting time.[2] The distribution of these waiting times is assumed to decay with a power-law exponent $\alpha < 1$.

An important addition to the fractional diffusion equation described in the previous section is the *reaction* mechanism, that corresponds to transactions between buy and sell orders that remove volume from the latent order book and set the transaction price. Within this framework, the density of buy $\phi_\mathrm{b}(x,t)$ and sell $\phi_\mathrm{s}(x,t)$ intentions at price x and time t solve the following set of coupled integro-differential equations in the reference frame of the "consensus" price:[3]

$$\partial_t \phi_\mathrm{b} = K_\alpha \mathfrak{D}_t^{1-\alpha}(\partial_{xx}\phi_\mathrm{b} - \nu\phi_\mathrm{b}) + \lambda\Theta(x_t - x) - R_\mathrm{sb}(x) \tag{6}$$
$$\partial_t \phi_\mathrm{s} = K_\alpha \mathfrak{D}_t^{1-\alpha}(\partial_{xx}\phi_\mathrm{s} - \nu\phi_\mathrm{s}) + \lambda\Theta(x - x_t) - R_\mathrm{sb}(x) \, , \tag{7}$$

where Θ denotes the Heaviside step function, and where $R_\mathrm{sb}(x)$ describes a reaction rate that instantaneously removes buy and sell "particles" as soon as they meet (see [19,20,21] for some work on fractional reaction-diffusion). Note that transactions remove exactly the same volume of buy and sell orders, justifying the fact that the same rate $R_\mathrm{sb}(x)$ appears in the two equations above. The term proportional to λ corresponds to an incoming flux of buy/sell intentions to the left/right of the transaction price x_t.

The non-linearity arising from the reaction term in the above equations can be abstracted by defining the combination $\psi(x,t) = \phi_\mathrm{b}(x,t) - \phi_\mathrm{s}(x,t)$, which precisely solves Eq. (2) with:

$$s(x,t) := \lambda\, \mathrm{sign}(x_t - x) \, , \tag{8}$$

and where the transaction price x_t is fixed by the condition:

$$\psi(x_t, t) = 0 \, . \tag{9}$$

The stationary order-book centred at $x_\infty = 0$ can be computed from Eqs. (4) and (8) as $\psi_\mathrm{eq}(x) = \lim_{t\to\infty}\psi(x,t)$ (see Appendix A for the expression of $\psi_\mathrm{eq}(x)$ in terms of hyper-geometric functions). In the vicinity of the transaction price, the stationary order book can be shown to be locally linear and its local shape is given by:

$$\psi_\mathrm{eq}(x) = -\mathcal{L}_\alpha x + O(x^2) \, , \tag{10}$$

where $\mathcal{L}_\alpha$ is given in Appendix A.

[2] Let us stress that price reassessments themselves are not fat-tailed distributed, *i.e.* intentions do not follow Levy flights. While such an extension would be interesting, it is not the point of the present paper and is left for future work.

[3] We here substantially simplify the discussion given in [10] where x_t in fact follows some additional exogenous dynamics, reflecting the evolution of the agents' expectations about the "consensus price". See [18] for an extended discussion.

4 Market impact

In this section we compute and analyze the impact of a meta-order with execution horizon T on the transaction price (see Appendix B for an alternative derivation based upon fractional diffusion with a moving boundary). Following Donier *et al.*, we introduce the meta-order of volume Q as an extra order flow that falls exactly at the transaction price such that the source term in Eq. (2) becomes ($t < T$):

$$s(x,t) = \lambda\,\mathrm{sign}(x_t - x) + m_t \delta(x - x_t)\ , \tag{11}$$

where m_t denotes the (possibly time dependent) execution rate, with $\int_0^T \mathrm{d}t\, m_t = Q$. The general solution as given by Eq. (4) with the source term of Eq. (11) reads:

$$\begin{aligned}\psi(x,t) &= [\mathcal{G}_\alpha * \phi_0](x,t) + \int_0^t \mathrm{d}u\, m_u \mathcal{G}_\alpha(x - x_u, t-u) \\ &+ \frac{i\lambda}{\pi} ⨍ \frac{\mathrm{d}k}{k} \int_0^\infty \mathrm{d}u\, F_\alpha\big[K(k^2+\nu)(t-u)^\alpha\big] e^{ik(x-x_u)}\ ,\end{aligned} \tag{12}$$

where $⨍$ denotes Cauchy's principal value.

In the following we "zoom" into the linear region of the book, close to the transaction price. More precisely, we consider the limit $\nu, \lambda \to 0$, while keeping $\mathcal{L}_\alpha \sim \lambda \nu^{(\alpha-2)/2\alpha}$ constant. In this limit – and starting from the equilibrium book $\psi_0(x) = \psi_{\mathrm{eq}}(x)$ – Eq. (12) becomes:

$$\psi(x,t) = -\mathcal{L}_\alpha x + \int_0^t \mathrm{d}u \frac{m_u}{\sqrt{4\pi K (t-u)^\alpha}} g_\alpha \left[\frac{(x-x_u)^2}{4K(t-u)^\alpha}\right] . \tag{13}$$

Making use of Eq. (9) yields the following self-consistent integral equation for the transaction price:

$$x_t = \frac{1}{\mathcal{L}_\alpha} \int_0^t \mathrm{d}u \frac{m_u}{\sqrt{4\pi K (t-u)^\alpha}} g_\alpha \left[\frac{(x_t-x_u)^2}{4K(t-u)^\alpha}\right] . \tag{14}$$

Provided that impact is small (or equivalently in the limit of small execution rates) one has $(x_t - x_u)^2 \ll 4K(t-u)^\alpha$, which recovers the propagator limit where the transaction price is linearly related to the order flow through a power-law decaying kernel:[4]

$$x_t = \frac{\sqrt{\pi}}{\mathcal{L}_\alpha \Gamma(1-\alpha/2)} \int_0^t \mathrm{d}u \frac{m_u}{\sqrt{4\pi K (t-u)^\alpha}}\ , \tag{15}$$

allowing us to identify the propagator decay exponent β with $\min(1/2, \alpha/2)$. Note that for $\alpha < 1$ the equality $\beta = (1-\gamma)/2$ can be achieved by the choice $\alpha = 1 - \gamma \in [0,1]$. Hence, the FLOB allows the price to be diffusive at all times in the presence of a persistent order flow. As mentioned in the introduction, real data suggests $\gamma \approx 0.5$ which implies $\alpha \approx 0.5$. For a constant execution rate $m_t = m_0 = Q/T$ and denoting $I_Q = x_T - x_0$ the impact of a meta-order of size Q, one obtains:

$$I_Q = \frac{Q^{1-\alpha/2}}{\mathcal{L}_\alpha} \frac{m_0^{\alpha/2}}{(2-\alpha)\Gamma(1-\alpha/2)\sqrt{K}}\ . \tag{16}$$

As one can see, Eq. (16) leads to $I_Q \sim Q^{0.75}$ for $\alpha = 0.5$, intermediate between a square-root and a linear behaviour. The pure square-root for small execution rates is only recovered in the limit $\alpha = 1$ considered by Donier *et al.* [10].

In the opposite limit however of fast execution – more precisely when $(x_t - x_u)^2 \gg 4K(t-u)^\alpha$ – one can show that the impact is again given by a square root law (see Appendix C):

$$I_Q = h(\alpha)\sqrt{\frac{2Q}{\mathcal{L}_\alpha}}\ , \tag{17}$$

where $h(\alpha) = (2-\alpha)^{-1/2} \leq 1$. Interestingly, the result is smaller than what a purely geometric argument would suggest, where the volume initially contained between x_0 and $x_T = x_0 + I_Q$ is executed against the incoming metaorder. In the latter case one can write:

$$Q = \int_0^{I_Q} \mathrm{d}x\, \mathcal{L}_\alpha x \quad \Rightarrow \quad I_Q = \sqrt{\frac{2Q}{\mathcal{L}_\alpha}}\ , \tag{18}$$

valid for $\alpha \geq 1$. When $\alpha < 1$, liquidity initially outside the interval $[x_0, x_T]$ manages to move inside that interval and meet the incoming metaorder, even in the fast execution limit. This provides more resistance to the metaorder, *i.e.* a slightly smaller impact.

The FLOB thus provides a framework in which concave impact is compatible with persistent order flow, although one expects a cross-over from a $Q^{1-\alpha/2}$ behaviour for "slow" execution to a $\sqrt{Q}$ behaviour for "fast" execution.[5]

The impact decay after the meta-order execution ($t > T$) can be computed by replacing the upper boundary of the integrals in Eqs. (14) and (15) by T. In the limit of small execution rates, one easily obtains:

$$\frac{I_Q(t > T)}{I_Q} = \left(\frac{t}{T}\right)^{1-\alpha/2} - \left(\frac{t-T}{T}\right)^{1-\alpha/2}\ , \tag{19}$$

which decays with an infinite slope for $t \to T^+$ and asymptotically equals $(1-\alpha/2)(t/T)^{-\alpha/2}$. In the limit of high execution rates, the impact decay for $t \gg T$ can be conveniently computed as in the slow execution limit and reads:

$$\frac{I_Q(t \gg T)}{I_Q} = \frac{m_0^{\alpha/2} Q^{(1-\alpha)/2}}{\sqrt{2(2-\alpha)K\mathcal{L}_\alpha}} \frac{1-\alpha/2}{\Gamma(1-\alpha/2)} \left(\frac{t}{T}\right)^{-\alpha/2}\ , \tag{20}$$

[4] Here, we have made use of the Taylor expansion of the function $g_\alpha(y)$ when $y \to 0$ (see [16,14]).

[5] On the definition of "slow" and "fast" for real markets where HFT significantly contribute, see [22].

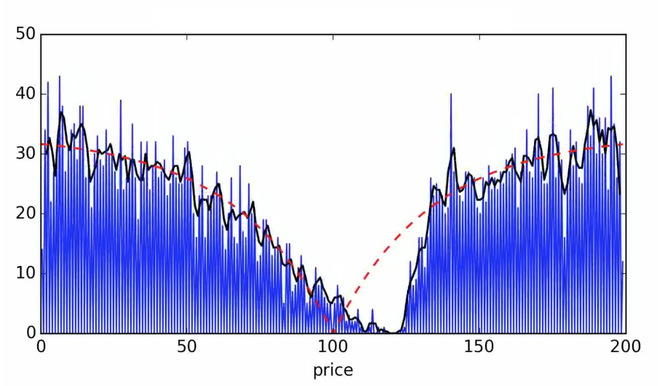

Fig. 1. Numerical simulation of the latent order book during a constant rate buy meta-order execution. The red dashed line corresponds to the equilibrium order book from which the simulation started at $t = 0$.

which only differs from the low execution result by the prefactor. For $t \to T^+$ the calculation is more subtle and relies on the source-free relaxation of the linearised order book profile in the vicinity of the price at the end of the metaorder execution (see [10]). One obtains:

$$\frac{I_Q(t \to T^+)}{I_Q} = 1 - \frac{\sqrt{K(t-T)^\alpha}}{I_Q} z^* \,, \tag{21}$$

where z^* solves: $a^- z - (a^+ - a^-) \int_z^\infty \mathrm{d}u \, \frac{u-z}{\sqrt{4\pi}} \, g_\alpha\left(\frac{u^2}{4}\right) = 0$ with $a^\pm = \lim_{\epsilon \to 0} \partial_x \psi(x_T \pm \epsilon, T)$.

5 Numerical simulation

In order to bolster our analytical results we performed a numerical simulation of the model. Because of the peculiar nature of fractional diffusion, such a simulation is more time consuming than a regular reaction-diffusion simulation. This is because time needs be continuous and each particle (order intention) must be treated independently. Each particle is labeled as *buy/sell*, and most importantly *next event time* drawn from a fat-tailed probability distribution function. In order to speed up the simulation we use a heap queue algorithm[6] to efficiently sort the up-coming events in time. The nature of the next event (diffusion or death) is drawn from a biased distribution depending on the diffusion and cancellation rates. Simultaneously a Poissonian rain of particles with rate λ falls into the book, each particle being naturally labeled with *buy/sell* depending on the side of the book it falls in. For the sake of simplicity the spread region between the best buy (bid) and the best sell (ask) is rain-free.

[6] Also called *priority queue algorithm*, the heap queue algorithm is a binary tree in which each parent node stores a value smaller than or equal to its children's, such that in our case the next event is always at the root. New events are pushed into the tree from the bottom and find their place in $\log N$ time [23].

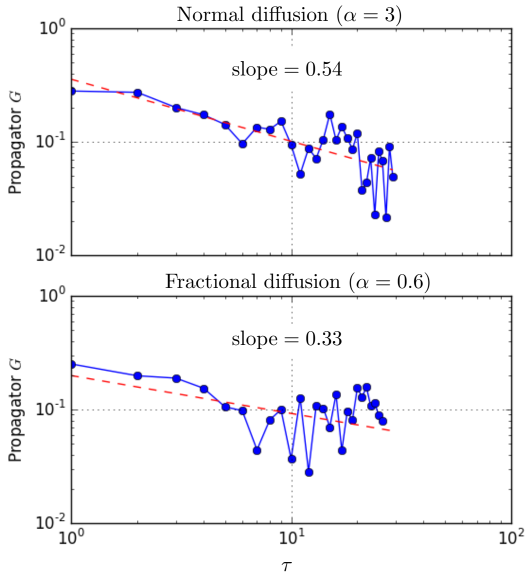

Fig. 2. Fit of the inverted propagator G_τ for a random order flux simulation: (top) in the normal diffusion case ($\alpha = 3 > 1$) with slope $0.54 \approx 1/2$, and (bottom) in the fractional diffusion case ($\alpha = 0.6 < 1$) with slope $0.33 \approx \alpha/2$.

For each potentially price changing event, namely each event involving a particle at the bid or ask, the heap is broken such that the relevant variables – volume profile, best bid and ask, spread, mid-price, event-time and deposition/cancellation/reaction rates - can be updated and stored. When a buy and a sell "particle" find themselves at the same spot, there are immediately removed from the book (reaction).

The simulation is initialised from the theoretical equilibrium situation, obtained after $t = 3\nu^{-1}$ in order to ensure a stationary state prior to the meta-order execution start. Reflective boundary conditions are implemented to ensure flat slopes at the (far away) edges of the book. A meta-order execution can be implemented through an additional rain of buy (resp. sell) particles that fall precisely at the best ask (resp. bid) with given rate m_t (see Fig. 1 for an illustration of a typical numerical experiment).

Using such a simulation to measure impact is not as straightforward as one could think. In the fast execution limit, discretisation induces the opening of a spread which struggles to refill during the execution. This effect is not included in the continuous reaction-diffusion model presented above. In particular, it is responsible for the fact that the impact decay is very difficult to reproduce within the numerical framework at hand. Furthermore, the slow execution limit is extremely time consuming, since one must ensure both a small execution rate, meaning T large, while still keeping and $T \ll \nu^{-1}$ for the above results to hold.[7]

[7] When $T \gg \nu^{-1}$, impact trivially becomes linear in Q.

We therefore use our numerical simulation to measure the propagator kernel in the small execution rate limit (see Eq. 15). To do so we replace the directional meta-order with a random, IID order flow $m_t = \varepsilon_t$ and follow the corresponding price changes. The discrete linear propagator G, defined as $p_t = \sum_{t'<t} G_{t-t'}\varepsilon_{t'}$ [11], can then be obtained by a linear regression and is plotted in Fig. 2. As one can see, in the normal diffusion case $\alpha > 1$ we obtain a square root decaying kernel as predicted by Donier *et al.* [10], while for the fractional diffusion case ($\alpha < 1$) we obtain a weaker power law decay (with exponent $\approx \alpha/2$) consistent with Eq. (15).

6 Conclusion

We have presented a fractional diffusion extension (FLOB) of the locally linear order book (LLOB) model of Donier *et al.* [10]. The fractional latent order book (FLOB) model presented here is motivated by the existence of a very broad spectrum of time scales in financial markets, spanning seconds to days or even weeks. This extended framework allows us to reconcile the long-memory nature of order flow with market efficiency (*i.e.* diffusive prices). The impact of a metaorder is a concave function of volume, crossing-over from a $Q^{0.75}$ behaviour for small execution rates to the usual $\sqrt{Q}$ behaviour for large execution rates. Another possibility to model agents heterogeneity is to keep a normal diffusion while introducing a wide spectrum of cancellation and deposition rates. Such an approach also yields very interesting, complementary results and is left for an up-coming communication [22].

Other possible generalisations can be considered. One is to introduce a broad distribution in the individual volumes of buy/sell orders. This is very relevant in view of the heterogeneity of asset sizes in the financial industry. However, the mathematical apparatus needed to investigate this case needs to be developed, even in standard diffusive case, since fluctuations in the volume profile $\phi(x,t)$ become dominant. Introducing fat-tailed jump lengths (Levy flights) would also be of interest. Indeed, the idea that agents' price reassessments can be large and sudden is actually quite realistic. Again, implementing this idea is not straightforward and will in particular need a careful analysis of the reaction terms, since the continuity of the price paths is lost.

In conclusion, we have proposed that the broad distribution of time scales in financial markets could be a crucial ingredient to model the complex dynamics of liquidity and allow one to reproduce realistic price dynamics in stylised Agent-Based Models.

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Appendix A: Equilibrium book

We here provide the shape of the equilibrium (meta-order free) order-book and its linear approximation in the vicinity of the price. The volume profile in the book $\psi_{\rm eq}$ as defined in Sect. 3 reads:

$$\psi_{\rm eq}(x) = \frac{i\lambda}{\alpha\pi K^{1/\alpha}} \int_0^\infty \mathrm{d}w\, \frac{F_\alpha(w)}{w^{1-1/\alpha}} \fint \mathrm{d}k\, \frac{e^{ikx}}{k(k^2+\nu)^{1/\alpha}}\ , \tag{22}$$

where:

$$\fint \mathrm{d}k\, \frac{e^{ikx}}{k(k^2+\nu)^{1/\alpha}} = c_1 x \cdot {}_1F_2\left[\left\{\tfrac{1}{2}\right\}, \left\{\tfrac{3}{2}, \tfrac{3}{2}-\tfrac{1}{\alpha}\right\}, \frac{\nu x^2}{4}\right] + c_2 x^{2/\alpha} \cdot {}_1F_2\left[\left\{\tfrac{1}{\alpha}\right\}, \left\{\tfrac{1}{2}+\tfrac{1}{\alpha}, 1+\tfrac{1}{\alpha}\right\}, \frac{\nu x^2}{4}\right], \tag{23}$$

where ${}_pF_q$ is the generalised hypergeometric function[8] [24] and where:

$$c_1 = i\nu^{(\alpha-2)/2\alpha}\sqrt{\pi}\,\Gamma\left(\tfrac{2-\alpha}{2\alpha}\right)/\Gamma\left(\tfrac{1}{\alpha}\right) \tag{24}$$

$$c_2 = -2i\Gamma\left(-\tfrac{2}{\alpha}\right)\sin\left(\tfrac{\pi}{\alpha}\right)\ . \tag{25}$$

Taylor expanding Eq. (22) at order 1 around $x = 0$ yields Eq. (10) with:

$$\mathcal{L}_\alpha = \frac{\lambda\nu^{(\alpha-2)/2\alpha}}{\alpha K^{1/\alpha}\sqrt{\pi}} \frac{\Gamma\left(\frac{2-\alpha}{2\alpha}\right)}{\Gamma\left(\frac{1}{\alpha}\right)} \int_0^\infty \mathrm{d}w\, \frac{F_\alpha(w)}{w^{1-1/\alpha}}\ . \tag{26}$$

Note that for normal diffusion – and using $F_1(w) = e^{-w}$ – one obtains $\mathcal{L}_1 = \lambda/K\sqrt{\nu}$ consistent with the results of Donier *et al.* [10].

Appendix B: Fractional diffusion with an absorbing moving boundary condition

We here present an alternative way to derive the FLOB results in the small execution rate limit. The idea is to first consider fractional diffusion with an absorbing moving boundary condition (at the price p_t) in perturbation theory, and then match the dynamics of p_t with the order flow.

Let us consider the source-less fractional diffusion equation together with an absorbing moving boundary condition at $x = f(t)$:

$$\partial_t\phi - K_\alpha \mathfrak{D}_t^{1-\alpha}\partial_{xx}\phi = 0 \tag{27}$$

$$\phi(x,t)\big|_{x=f(t)} = 0\ . \tag{28}$$

In the frame of reference of the absorbing boundary, the problem reads:

$$\left[\partial_t - \dot{f}\partial_x\right]\phi(x,t) - K\left[\partial_t - \dot{f}\partial_x\right]\int_0^t du\, \frac{\partial_{xx}\phi(x+f(t)-f(u),u)}{(t-u)^{1-\alpha}} = 0 \tag{29}$$

$$\phi(x,t)\big|_{x=0} = 0\ , \tag{30}$$

[8] ${}_pF_q\left[\{a_1,\dots,a_p\},\{b_1,\dots,b_q\},u\right] = \sum_{n=0}^\infty \frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\frac{u^n}{n!}$, where $(.)_n$ denotes the Pochhammer symbol for the rising factorial [24].

Expanding the integrand of Eq. (29) at first order for a slow moving boundary $\phi(x+f(t)-f(u),u) \simeq \phi(x,u) + (f(t)-f(u))\,\partial_x\phi(x,u)$ yields:

$$\partial_t\phi - K_\alpha \mathfrak{D}_t^{1-\alpha}\partial_{xx}\phi = h\big(\phi(x,t)\big) \ , \tag{31}$$

where at first order $h\big(\phi(x,t)\big) =$

$$\dot f \partial_x\phi(x,t) - K(1-\alpha)\int_0^t du\, \frac{\partial_{xxx}\phi(x,u)}{(t-u)^{1-\alpha}}\frac{f(t)-f(u)}{t-u} \ . \tag{32}$$

Within first order perturbation theory we let $g(\phi) \to \epsilon g(\phi)$ and $\phi = \phi_0 + \epsilon\phi_1$ into Eq. (31). This yields for the ground state:

$$\partial_t\phi_0 - K_\alpha \mathfrak{D}_t^{1-\alpha}\partial_{xx}\phi_0 = 0 \tag{33}$$

$$\phi_0(x,t)\big|_{x=0} = 0 \ , \tag{34}$$

and for the first order perturbation:

$$\partial_t\phi_1 - K_\alpha \mathfrak{D}_t^{1-\alpha}\partial_{xx}\phi_1 = h\big(\phi_0(x,t)\big) \tag{35}$$

$$\phi_1(x,t)\big|_{x=0} = 0 \ . \tag{36}$$

In addition, we take the initial condition to be given by $\phi_{\rm eq}(x) = -\mathcal{L}_\alpha x$ such that:

$$\phi_0(x,t)\big|_{t=0} = -\mathcal{L}_\alpha x \tag{37}$$

$$\phi_1(x,t)\big|_{t=0} = 0 \ . \tag{38}$$

Noting that $h\big(\phi_0(x,t)\big) = -\mathcal{L}_\alpha \dot f(t)$ and making use of the method of mirror images yields:

$$\phi(x,t) = -\mathcal{L}_\alpha x + \epsilon\mathcal{L}_\alpha \mathrm{sign}(x)\int_0^t dt' \dot f(t')\int_0^\infty dx' [...] \ , \tag{39}$$

where $[...] = \mathcal{G}_\alpha(x-x',t-t') - \mathcal{G}_\alpha(x+x',t-t')$, and where the kernel $\mathcal{G}_\alpha$ is given in Eq. (5). In order to obtain a closed-form solution, one needs to add the condition that the diffusion current discontinuity at the origin, namely the break of slope $\mathcal{J}_\alpha|_{x=0^+} - \mathcal{J}_\alpha|_{x=0^-}$ where $\mathcal{J}_\alpha = -K_\alpha \mathcal{D}_t^{1-\alpha}\partial_x\phi$, must match the order flow of the meta-order (with constant execution rate m_0):

$$\lim_{\epsilon\to 0} K_\alpha \Big|\mathcal{D}_t^{1-\alpha}\big(\partial_x\phi|_{x=\epsilon} - \partial_x\phi|_{x=-\epsilon}\big)\Big| = m_0 \ . \tag{40}$$

Letting $f(t) = p_t = Bt^\mu$ into Eq. (40), together with Eqs. (39) and (5) enforces $\mu = 1-\alpha/2$, consistent with Eq. (16).

Appendix C: Price impact in the fast execution limit

We here compute the price impact in the limit of large execution rates as given by Eq. (17). Letting $x_t = A\sqrt{t}$ into Eq. (14) with a constant execution rate $m_u = m_0$ and changing variables through $u = t(1-v)$ yields:

$$A\sqrt{t} = \frac{t^{1-\alpha/2}}{\mathcal{L}_\alpha}\int_0^1 \mathrm{d}v\, \frac{m_0}{\sqrt{4\pi K v^\alpha}}\, g_\alpha\Big[\tfrac{A^2 t^{1-\alpha}}{4K}\tfrac{\left(1-\sqrt{1-v}\right)^2}{v^\alpha}\Big] \ . \tag{41}$$

Noting that the integrand dominates for $v\to 0$ – which is $(1-\sqrt{1-v})^2 v^{-\alpha} \simeq v^{2-\alpha}/4$ – and letting $z = vt^\delta$ where $\delta = (1-\alpha)/(2-\alpha)$, one obtains:

$$A = \frac{1}{\mathcal{L}_\alpha}\int_0^{t^\delta} \mathrm{d}z\, \frac{m_0}{\sqrt{4\pi K z^\alpha}}\, g_\alpha\left[\frac{A^2 z^{2-\alpha}}{16K}\right] \ . \tag{42}$$

Letting $w = A^2 z^{2-\alpha}/(16K)$ in the limit $A\to\infty$ yields:

$$A^2 = \frac{2m_0}{\mathcal{L}_\alpha}\frac{1}{2-\alpha}\frac{1}{\sqrt{\pi}}\int_0^\infty \mathrm{d}w\, \frac{g_\alpha[w]}{\sqrt{w}} \ , \tag{43}$$

and making use of the normalisation of the fractional diffusion kernel $\int \mathrm{d}x\, \mathcal{G}_\alpha(x,t) = 1$, which together with Eq. (5) can also be written as $\int_0^\infty \mathrm{d}w\, g_\alpha[w]/\sqrt{w} = \sqrt{\pi}$, finally yields Eq. (17).

References

1. Grinold R C and Kahn R N 2000 *Active Portfolio Management* (McGraw Hill New York)
2. Almgren R, Thum C, Hauptmann E and Li H 2005 *Risk* **18** 58–62
3. Tóth B, Lemperiere Y, Deremble C, De Lataillade J, Kockelkoren J and Bouchaud J P 2011 *Physical Review X* **1** 021006
4. Mastromatteo I, Tóth B and Bouchaud J P 2014 *Physical Review E* **89** 042805
5. Donier J and Bonart J 2015 *Market Microstructure and Liquidity* **1** 1550008
6. Zarinelli E, Treccani M, Farmer J D and Lillo F 2015 *Market Microstructure and Liquidity* **1** 1550004
7. Gabaix X, Gopikrishnan P, Plerou V and Stanley H E 2003 *Nature* **423** 267–270
8. Farmer J, Gerig A, Lillo F and Waelbroeck H 2013 *Quant. Finance* **13** 1743–1758
9. Mastromatteo I, Tóth B and Bouchaud J P 2014 *Phys. Rev. Lett.* **113**
10. Donier J, Bonart J F, Mastromatteo I and Bouchaud J P 2015 *Quantitative Finance* **15** 1109–1121
11. Bouchaud J P, Gefen Y, Potters M and Wyart M 2004 *Quantitative Finance* **4** 176–190
12. Lillo F and Farmer J D 2004 *Studies in Nonlinear Dynamics & Econometrics* **8** 1
13. Bouchaud J P, Farmer J D and Lillo F 2008 *How markets slowly digest changes in supply and demand* (Elsevier: Academic Press)
14. Metzler R and Klafter J 2000 *Physics reports* **339** 1–77
15. Metzler R and Nonnenmacher T F 2002 *Chemical Physics* **284** 67–90
16. Schneider W R 1990 *Fractional diffusion. In Dynamics and Stochastic Processes Theory and Applications* (Springer Berlin Heidelberg)
17. Henry B I and Wearne S L 2000 *Physica A: Statistical Mechanics and its Applications* **276** 448–455
18. Donier J and Bouchaud J P 2016 *Journal of Statistical Mechanics: Theory and Experiment* **2016** 123406
19. Henry B I and Wearne S L 2000 *Physica A: Statistical Mechanics and its Applications* **276** 448–455
20. Henry B I, Langlands T A M and Wearne S L 2006 *Physical Review E* **74** 031116

21. Yuste S B, Acedo L and Lindenberg K 2004 *Physical Review E* **69** 036126
22. Benzaquen M and Bouchaud J P 2017 *In preparation*
23. Brown M R 1978 *SIAM journal on computing* **7** 298–319
24. Abramowitz M and Stegun I A 1964 *Handbook of mathematical functions* vol 55 (Courier Corporation)