We've walked a million miles for one of these smiles

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Abstract

We derive a new, exact and transparent expansion for option smiles, which lends itself both to analytical approximation and to congenial numerical treatments. We show that the skew and the curvature of the smile can be computed as exotic options, for which the Hedged Monte Carlo method is particularly well suited. When applied to options on the S&P index, we find that the skew and the curvature of the smile are very poorly reproduced by the standard Edgeworth (cumulant) expansion. Most notably, the relation between the skew and the skewness is inverted at small and large vols, a feature that none of the models studied so far is able to reproduce. Furthermore, the around-the-money curvature of the smile is found to be very small, in stark contrast with the highly kurtic nature of the returns.

Understanding the shape of volatility smiles in option markets is arguably one of the most active field of research in quantitative finance [1, 2]. By definition, the existence of an option smile is the sign that the standard Black-Scholes model is not an adequate representation of the stochastic dynamics of financial assets. A huge variety of models have been proposed over the years in order to account for the non-Gaussian nature of price changes and the corresponding option smiles: jumps and Lévy processes [3], GARCH and stochastic volatility models [4] (including the popular Heston and SABR models [5]), multiscale (multifractal) models [6, 7], mixed jumps/stochastic vol. models, etc. Another model-free strand of research that sheds light on the origin and the general structure of option smiles, is to assume that the corrections to the Black-Scholes model can somehow be considered as "small". A natural idea is to use a Edgeworth cumulant expansion of the distribution of the price change $r_T = \frac{S_T - S_0}{S_0}$ (see Appendix). Working with the Bachelier model as benchmark (i.e. $(S_t)_{t \geq 0}$ follows a Brownian motion instead of a Geometric Brownian motion), the authors of [8, 9] have obtained the following cumulant expansion for the smile:

$$\sigma_{BS} = \sigma(1 + \frac{\mathcal{S}_T}{6}\mathcal{M} + \frac{\kappa_T}{24}(\mathcal{M}^2 - 1) + \dots), \qquad \mathcal{M} := \frac{K - S_0}{S_0 \sigma \sqrt{T}} \tag{1}$$

where $\sigma\sqrt{T}$ denotes the standard deviation of r_T, $\mathcal{M}$ is the rescaled moneyness of the option, and $\mathcal{S}_T \ll 1$ and $\kappa_T \ll 1$ are respectively the skewness and kurtosis of r_T, which are assumed to be small for the expansion to make sense.[1] Note that the authors of [10] independently obtained a similar formula within a Black Scholes context, around the same time. The two formulas coincide for $\sigma\sqrt{T} \ll 1$, which is expected because in that limit the Black-Scholes model becomes identical to the Bachelier model.

The interest of the above formula is that it allows one to understand why smiles have generically an asymmetric parabolic shape, with an asymmetry related to the skewness of the distribution of the underlying, while the curvature of the smile is proportional to its kurtosis. Since from general arguments the distribution of r_T becomes Gaussian at large times (albeit perhaps very slowly due to long-memory effects [9]), the asymmetry and curvature of the smile are expected to go down with maturity T, as indeed observed in option markets.

However, the expansion (1) involves moments up to order 4 and as such is troublesome both theoretically and practically. Indeed, the cumulative distribution of returns $P(|r_T| > x)$

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[1] Eq. (1) in fact assumes that $\mathcal{S}_T^2 \ll \kappa_T$, which is often the case in practice.

is found empirically to decrease as $x^{-\mu}$ with $\mu \approx 3$ for many assets (see e.g. [11, 12]) and thus the moment of order 3 is formally divergent. At any rate, it is in practice very difficult to estimate moments of order 3 and 4 because of the noise induced by large events. One possibility is to replace the skewness and kurtosis of r_T by some lower moment approximations (cf. [9]). However, this procedure is ambiguous as there is freedom in the choice of lower moments to be used.

The purpose of this note is to establish a different smile expansion formula, which is general, rigorous and involves no moments of order greater than 2. This new smile formula lends itself to analytical treatments, which allow one to recover, for example, the recent results of Bergomi & Guyon [13]. But perhaps more importantly, our formula can be coupled to the "Hedged Monte-Carlo Method" of [14, 15] to yield a powerful numerical method, which provides accurate estimates of the smile parameter for arbitrarily complex models of the underlying. One may even use historical data directly, short-circuiting any modelling assumption. Our formula, derived in the Appendix, reads:

$$\sigma_{BS} = \sigma(\alpha_T + \beta_T \mathcal{M} + \gamma_T \mathcal{M}^2 + O(\mathcal{M}^3)), \tag{2}$$

with the coefficients $\alpha_T, \beta_T, \gamma_T$ given by:

$$\alpha_T = \sqrt{\frac{\pi}{2}} E[|u_T|], \quad \beta_T = \sqrt{\frac{\pi}{2}} \left[1 - 2P(u_T > 0)\right], \quad \gamma_T = \sqrt{\frac{\pi}{2}} p_T(0) - \frac{1}{2\alpha_T}, \tag{3}$$

where $u_T = r_T/\sigma\sqrt{T}$ and $p_T(.)$ is the density of u_T. Using a Edgeworth expansion, one can show that both smile formulae (1) and (2) coincide in the limit where cumulants are small and cumulants higher than four can be neglected (see Appendix). One finds in particular $\beta_T \approx \mathcal{S}_T/6$. But what is remarkable is that while the "old" smile formula (1) is highly sensitive to extreme events, the new formula (2) only involves low moments of the distribution of u_T. In particular, the skew of the smile, as measured by the coefficient β_T, is technically a moment of order 0, i.e. the large events do not play any role at all.

This new smile formula can be used in different ways. One can for example obtain directly the coefficients $\alpha_T, \beta_T, \gamma_T$ in a vol-of-vol expansion, recovering the Bergomi-Guyon results to lowest order [16]. One of the salient results of Bergomi & Guyon [13] is that to leading order in vol-of-vol and for a broad family of linear models, the Edgeworth expansion result $\beta_T = \mathcal{S}_T/6$ holds, i.e. the *skew* of the smile β, and the *skewness* $\mathcal{S}_T$ of the distribution of u_T, are very simply related. A first exercise is to use our formula to investigate the case where

the correlation between stock returns and volatility is non-linear. Assume for example a stochastic volatility model where $r_T = \int_0^T \sigma_t dW_t$, with dW_t the standard Wiener process, and where the instantaneous volatility σ_t^2 is given by:

$$\sigma_t^2 = \overline{\sigma}^2 \left[1 + 2\epsilon(|\xi_t + \theta| \, 1_{\xi_t + \theta < 0})\right], \qquad \xi_t = \omega \int_0^t e^{-\omega(t-u)} dW_u, \quad \epsilon > 0. \tag{4}$$

Here $\overline{\sigma}^2$ is a "baseline" volatility (which might itself evolve over long times scales, see below).

When the threshold θ is zero, the above equation means that when recent returns are negative, the future volatility will be higher than usual (this is the standard leverage effect [9, 17]), while when the recent returns are positive, there is no impact on the future vol. When $\theta > 0$, only relatively large negative returns will increase the volatility, while even small positive returns increase the volatility when $\theta < 0$. The coefficients β_T and $\mathcal{S}_T$ can be computed to first order in ϵ in this case [16]. One finds that the equality $\beta_T = \mathcal{S}_T/6$ only holds when $\theta = 0$. When $\theta > 0$, one the other hand, the skew amplitude $|\beta_T|$ is *smaller* than $|\mathcal{S}_T|/6$, and vice-versa when $\theta < 0$. This shows that even for small vol-of-vols, non linear effects can significantly affect the relation between skew and skewness, and that there can be no general link between the two.

Although analytical results are interesting, we believe that the true added-value of our new smile expansion comes from the following remark: the three coefficents α_T, β_T and γ_T can be interpreted as the *average payoff of some "exotic" options*. Indeed, α_T is simply an at-the-money straddle, β_T is related to an at-the-money binary option, and the first term of γ_T is a "no move" option which pays if the underlying ends very close to its initial price. [2] Having interpreted these coefficients as option prices, one can use the "Hedged Monte-Carlo" (HMC) method proposed in [14, 15] to price these options numerically. This is interesting for at least two reasons (see [9, 14, 15, 18] for a more thorough discussion): first, the HMC has by construction a low variance, that allows one to price options accurately with a relatively small number of paths; second, the P&L of the hedge automatically transforms historical or empirical probabilities into risk-neutral ones. Technically, the pricing of the "exotic" options is done in a way very similar to what is described in [14, 15], except that we do not try here to determine the optimal hedge for these options, but rather use a (sub-optimal)

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[2] It is in fact common folklore, inherited from Eq. (1), that buying slightly out of the money and selling slightly in the money allows one to bet on the "skewness", while a butterfly trade allows one to bet on the "kurtosis".

Black-Scholes hedge. This reduces HMC to a standard reduced-variance Monte-Carlo [19] which is conceptually much simpler and faster numerically. Although only approximate, this pruned down version of HMC is accurate enough for the present purpose. The other subtle point concerns the pricing of the "no move" option, since one has to introduce a window with a small but finite width δ, and carefully extrapolate the results to $\delta = 0$. We in fact use a Gaussian payoff function, $\exp(-u_T^2/2\delta^2)/\sqrt{2\pi\delta^2}$ for that purpose. We have calibrated the method on known cases – for example on the above non-linear model, with good agreement between the analytical calculations and the HMC results.

We now apply our method to real time series, using historical trajectories to price our exotic options and determine directly the skew β_T and curvature γ_T of the smile, without any a priori model. The data we use is the time series of daily returns (open-high-low-close) of the S&P500 index in the period $1/1/1970 - 31/12/2011$. We divide the sample into two bins: one corresponding to high volatilities (larger than the median), the other to low volatilities, where the volatility is a 20 day exponential moving average of a Rogers-Satchell estimator of the squared daily volatility *before* the day the smile coefficients are determined. We show in Fig. 1 the "fair" skew β_T as a function of T for options between 1 and 20 days, that we compare with the prediction $\mathcal{S}_T/6$ based on a direct measure of the third moment skewness of the distribution of u_T. We find, perhaps surprisingly, that while $|\mathcal{S}_T|/6$ is systematically larger than the skew $|\beta_T|$ for large volatilities, the opposite is true for low volatilities. Let us emphasize here that we are not speaking of implied volatility smiles from option markets, but rather of theoretical predictions of what the *fair* parameters of the smile should be. Including some information from option markets could be done along the lines of [20].

In Fig. 2, we show the smile curvature γ_T as a function of T for the same two volatility regimes above, and compare it to $\kappa_T/24$, the curvature obtained from the Edgeworth expansion (1). Here again the results are surprising: we find that the "fair" curvature γ_T of the smile around the money is close to zero, both for high and low volatilities. This is very different from what the empirical kurtosis of the distribution suggests, especially in the high vol regime where the kurtosis of returns is empirically very substantial, in agreement with the fact, noted above, that the distribution of returns exhibits power-law tails. [3] This is a striking illustration of the misleading character of the Edgeworth expansion (1): the

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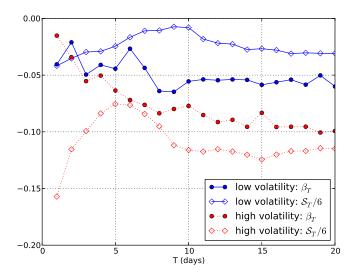[3] The kurtosis might even be mathematically divergent.

FIG. 1: S&P500 data. Skew β_T and skewness $\mathcal{S}_T/6$ for different volatility regimes. Note the inversion of the order between the two quantities as the volatility increases.
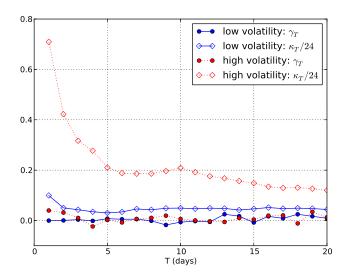

FIG. 2: S&P500 data. Curvature γ_T and kurtosis $\kappa_T/24$ for different volatility regimes. Note that, surprisingly, the curvature of the smile is found to be very small in both volatility regimes.

around-the-money smile of index options should be close to a straight line with very little curvature, as indeed often seen on implied smiles, whereas the Edgeworth expansion would predict highly curved smiles.

The results on the skew of the smile shown in Fig. 1 are clearly inconsistent with the

relation $\beta_T = \mathcal{S}_T/6$, and cannot be understood either within the above non-linear leverage effect with a fixed parameter θ. A way out would be to assume that θ itself depends on the "baseline" volatility $\overline{\sigma}$ in Eq. (4) above, with $\theta > 0$ during high volatility periods and $\theta < 0$ during low volatility periods. This could be interpreted as meaning that at high vols, a significant down-trend is required to drive the volatility even higher, whereas at low vols, it takes a moderate down-trend to move the volatility up. It is also instructive to compare these results with the predictions of a popular asymmetric GARCH model used to model the dynamics of the S&P500 index [21]. Writing the returs as $\sigma_t \eta_t$ with $\eta_t \sim \mathcal{N}(0,1)$, the dynamics of σ_t is postulated to be:

$$\sigma_t = \sigma(1 + \chi_t); \qquad \chi_{t+1} = \rho \chi_t + \nu(1 + \chi_t)\left[\eta_t^2 1_{\eta_t < 0} - \frac{1}{2}\right]. \tag{5}$$

We have run our HMC on synthetic time series generated using the asymmetric GARCH (GAARCH) model with parameters $\rho = 0.9$ and $\nu = 0.1$, corresponding to a "memory time" of the volatility equal to 10 days. One can of course now generate as many trajectories as needed to get good statistics. The results for β_T and γ_T together with their corresponding cumulant expansion expressions are reported in Fig. 3 for the high volatility regime. We observe in this case a trend similar to the S&P500 data: the skewness $|\mathcal{S}_T|/6$ exceeds $|\beta_T|$ in absolute value and $\kappa_T/24$ is systematically larger than γ_T, although the latter is significantly different from zero. However, for low volatilities (not shown) the behavior of the skew is qualitativaly different from the S&P500 data. In that case β_T and $\mathcal{S}_T/6$ are very close (except for very small T), but with still $|\mathcal{S}_T|/6 > |\beta_T|$. In other words, the inversion observed on empirical data for low vols cannot be reproduced within the GAARCH model. That $|\mathcal{S}_T|/6 > |\beta_T|$ can in fact be checked analytically in a small ν expansion. Note this inequality implies that the "Skew Stickiness Ratio" introduced by Bergomi in [22] is expected to be smaller than 2 at small maturities.

As a conclusion, we have set up a new, exact and transparent expansion for option smiles, which lends itself both to analytical approximation and, perhaps more importantly, to congenial numerical treatments. We have shown that the skew and the curvature of the smile can be computed as exotic options, for which the Hedged Monte Carlo method is particularly well suited. When applied to options on the S&P index, we have found that the skew and the curvature of the smile are very poorly reproduced by the standard Edgeworth (cumulant) expansion, or by the Bergomi-Guyon vol-of-vol expansion (at least to lowest
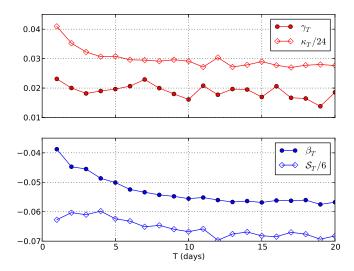
FIG. 3: GAARCH synthetic data from equation (5), with $\rho = 0.9$, $\nu = 0.1$, and 10^6 Monte Carlo paths for high volatility regime. Upper panel: curvature γ_T and kurtosis $\kappa_T/24$. Lower panel: skew β_T and skewness $\mathcal{S}_T/6$.

order). Most notably, the relation between the skew and the skewness is inverted at small and large vols, a feature that none of the models studied so far is able to reproduce. We have argued that some coupling between the leverage effect and the volatility level is needed to capture such non-trivial statistical features. Finally, the around-the-money curvature of the smile is found to be very small, in stark contrast with the highly kurtic nature of the returns. This would also require some more detailed understanding. We hope that the ideas and methods presented in this paper are of general interest, and provide a fruitful framework to interpret the information provided by option markets.

Technical Appendix

Proof of the smile formula

We denote the price $S_T = S_0(1 + r_T)$. We suppose that r_T is centered and that $\sigma^2 T = E[r_T^2]$. We set $u_T = r_T/\sigma\sqrt{T}$. We have:

$$E[(S_T - K)_+] = S_0 E[(r_T - (\frac{K}{S_0} - 1))_+] = S_0 \int_{\frac{K-S_0}{S_0}}^{\infty} P(r_T > x)dx$$

$$= S_0\sigma\sqrt{T} \int_{\mathcal{M}}^{\infty} P(u_T > u)du$$

where we introduce the renormalized moneyness $\mathcal{M} = \frac{K-S_0}{\sigma\sqrt{T}S_0}$. A direct expansion in $\mathcal{M}$ leads to:

$$E[(S_T - K)_+] \approx S_0\sigma\sqrt{T}(\frac{E[|u_T|]}{2} - \mathcal{M}P(u_T > 0) + \frac{\mathcal{M}^2}{2}p_T(0))$$

In the Gaussian case, we get therefore:

$$E[(S_T - K)_+] \approx S_0\sigma_{BS}\sqrt{T}(\frac{1}{\sqrt{2\pi}} - \frac{\mathcal{M}_G}{2} + \frac{\mathcal{M}_G^2}{2}\frac{1}{\sqrt{2\pi}})$$

where $\mathcal{M}_G = \frac{K-S_0}{\sigma_{BS}\sqrt{T}S_0}$ is the Gaussian moneyness.

We make now make the assumption:

$$\sigma_{BS} = \sigma(\alpha' + \beta'\mathcal{M}_G + \gamma'\mathcal{M}_G^2), \tag{6}$$

so by definition $\mathcal{M} = \mathcal{M}_G(\alpha' + \beta'\mathcal{M}_G + \gamma'\mathcal{M}_G^2)$. The smile formula corresponds to the following identity:

$$\sigma_{BS}(\frac{1}{\sqrt{2\pi}} - \frac{\mathcal{M}_G}{2} + \frac{\mathcal{M}_G^2}{2}\frac{1}{\sqrt{2\pi}}) = \sigma(\frac{E[|u_T|]}{2} - \mathcal{M}P(u_T > 0) + \frac{\mathcal{M}^2}{2}p_T(0))$$

Therefore, we get the following equation (to the order $\mathcal{M}_G^2$):

$$(\alpha'+\beta'\mathcal{M}_G+\gamma'\mathcal{M}_G^2)(\frac{1}{\sqrt{2\pi}}-\frac{\mathcal{M}_G}{2}+\frac{\mathcal{M}_G^2}{2}\frac{1}{\sqrt{2\pi}}) = (\frac{E[|u_T|]}{2}-(\alpha'\mathcal{M}_G+\beta'\mathcal{M}_G^2)P(u_T > 0)+\frac{\alpha'^2\mathcal{M}_G^2}{2}p_T(0)).$$

which finally lead to the following identification:

$$\begin{cases} \alpha' = \sqrt{\frac{\pi}{2}}E[|u_T|] \\ \beta' = \pi E[|u_T|](\frac{1}{2} - P(u_T > 0)) \\ \gamma' = \sqrt{2\pi}(\frac{\alpha'^2 p_T(0)}{2} - \frac{\alpha'}{2\sqrt{2\pi}} + \pi E[|u_T|](\frac{1}{2} - P(u_T > 0))^2) \end{cases}$$

Switching back to standard moneyness and using $\sigma_{BS} = \sigma(\alpha + \beta\mathcal{M} + \gamma\mathcal{M}^2)$ finally leads to:

$$
\begin{cases}
\alpha = \sqrt{\frac{\pi}{2}}E[|u_t|] \\
\beta = \frac{\beta'}{\alpha} = \sqrt{2\pi}(\frac{1}{2} - P(u_t > 0)) \\
\gamma = \frac{\gamma'}{\alpha^2} - \frac{\beta'^2}{\alpha^3} = \sqrt{2\pi}(\frac{p_T(0)}{2} - \frac{1}{2\alpha\sqrt{2\pi}})
\end{cases}
$$

The Edgeworth expansion

Assuming $\mathcal{S}_T, \kappa_T \ll 1$, the Edgeworth expansion reads:

$$
P(\frac{r_T}{\sigma\sqrt{T}} > x) - \overline{N}(x) \approx \frac{\mathcal{S}}{6}N^{(3)}(x) - \frac{\kappa}{24}N^{(4)}(x).
$$

where $N(x) = \int_{-\infty}^{x} \frac{e^{-t^2/2}}{\sqrt{2\pi}}dt$, $\overline{N} = 1 - N$ and $N^{(n)}$ refers to the nth derivative of N. This leads to the following approximations:

$$
\begin{cases}
\alpha \approx 1 - \frac{\kappa_T}{24} \\
\frac{1}{2} - P(u_T > 0) \approx -M_T p_T(0) \approx -\frac{M_T}{\sqrt{2\pi}} \ll 1 \\
p_T(0) - \frac{1}{\sqrt{2\pi}} \approx \frac{1}{\sqrt{2\pi}}\frac{\kappa_T}{8} \ll 1
\end{cases}
$$

where M_T is the median of u_T. With these assumptions, we get the following expressions for α, β, γ:

$$
\begin{cases}
\alpha \approx 1 - \frac{\kappa_T}{24} \\
\beta \approx -M_T \\
\gamma \approx \frac{\sqrt{2\pi}}{2}(\frac{1}{\sqrt{2\pi}}(1 + \frac{\kappa_T}{8}) - \frac{1}{\sqrt{2\pi}}(1 + \frac{\kappa_T}{24})) \approx \frac{\kappa_T}{24}
\end{cases}
$$

Within the aforementioned approximation, both smile formulas coincide since we know that the Edgeworth expansion also leads to:

$$
\mathcal{S}_T \approx -6M_T, \quad \kappa_T \approx 24(1 - \sqrt{\frac{\pi}{2}}E[|u_T|])
$$

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