Welcome to a non-Black

Jean-Philippe Bouchaud and *Marc Potters*, citing option markets and risk awareness, challenge the view that the Black–Scholes model needs little improvement—in fact, it should be seen as a special case of a more general theory.

In the last issue of *Quantitative Finance*, Jessica James argues that there is 'very little need for improvements' over the Black–Scholes model. It is true that the market has already corrected for all the inadequacies and oversimplifications of the Black–Scholes model with historical volatility. The model is instead used with an adequately modified 'implied' volatility which is both strike and maturity dependent. But this procedure is both theoretically inconsistent and somewhat awkward (the volatility already becomes a two-dimensional sheet for simple options and a three-dimensional cube for swaptions!). But, we are told, markets have become so used to handling this object that we shouldn't bother trying to change it. The same type of argument was used when Kepler proposed that the orbits of planets were ellipses rather than Ptolemy's epicycles. The prediction of the planets' positions using epicycles had become so precise (after centuries of fixes and stitches to improve it) that at first the new theory was much less predictive! But not only was it more satisfying from an intellectual point of view, it opened new paths, posed new questions, and thereby led to considerable progress—both fundamental and applied (anyone for telecommunication satellites on geostationary epicycles?).

We believe that to some extent the same applies to Black–Scholes. True, the theory is mathematically beautiful, but somewhat counter-intuitive and often misleading. The cornerstone of its derivation, Ito's lemma, is deceptively simple and sweeps all the subtleties under the rug. More embarrassingly, everything that's really interesting about options wouldn't even exist in a Black–Scholes world. As we argue now, improvements over the Black–Scholes model are needed, not only from an intellectual and pedagogical standpoint, but also because of the many exciting (and sometimes unforseeable) operational developments in option trading, hedging and risk management that are valuable in an ever more competitive environment.

The most remarkable aspect of the Black–Scholes model is the possibility of finding a perfect *riskless* hedging strategy, which allows the writer of an option to know his P&L with certainty. The price of the option is then fixed so as to make this P&L zero—not zero on average, but zero plus or minus nought! This is the *only* possibility since otherwise either the writer or the buyer of the option would lose money *for sure*. This surprising zero-risk property has its roots in the two basic assumptions of the Black–Scholes model: the price increments are Gaussian random variables on every (arbitrarily small) time scale *and* the hedging strategy re-balancing takes place infinitely fast. These assumptions allow one to use stochastic differential calculus. Using Ito's magic wand, the zero-risk property and the Black–Scholes price formula are obtained in a nutshell. However, as soon as the price increments are non-Gaussian (for example if the volatility itself is stochastic) or if the hedging time is non-zero, the risk associated to option trading can no longer be made to vanish. The zero-risk property is a miracle, an exception rather than the rule. If the risk is non-zero, then the price is not known for sure, 'expensive' or 'cheap' will now depend on the perception of risk of each market participant. But it is this fundamental uncertainty on the price that allows for the very existence of option markets, where the risk can be traded. No risk, no market.

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This residual risk is therefore important to compute, not only because it will be partially reflected in the price (and therefore in the implied volatility) but also for risk management purposes. The fact that the P&L becomes uncertain raises new questions, which are meaningless in the Black–Scholes model, but crucial for risk management. Is the P&L distribution skewed? (Yes, negatively for the writer.) Does it have fat tails? (Yes.) What should one minimize: the P&L volatility, or its value-at-risk? (Depends on who you are, the risk management directives, etc.) But different risk objectives lead to different hedging strategies, and therefore different prices. So what is the risk in reality? Black–Scholes says zero, but zero is not a good approximation of anything. The answer is, for a typical one month at-the-money option, that the residual risk is 30% of the option premium (see figure 1)—not small by any means. What is the risk of a full option book, composed of many different options? (Black–Scholes says: $0 + 0 = 0$.) Is the optimal hedge for the whole book the simple linear superposition of all individual optimal hedges? These questions very seldom appear in textbooks on option theory, and no wonder. We believe option theory should rather be taught starting from generic models, which highlight the need to hedge but the impossibility of *perfect* hedges, leaving Black–Scholes as very special case of a more general theory. Of course one loses the neat idea of unique prices, but gains the existence of option markets, and the awareness of risk—quite useful indeed for future traders and quants.

Let us now turn to a more interesting aspect of post-Black–Scholes models that Jessica James discussed in her paper: can these be used to *understand* and therefore *predict* option prices? Markets give us their own prediction of the option prices; and therefore, says James, we don't really need any theory since we just have to *observe* the price. But the markets can be off, in the sense that an accurate theory

–Scholes world

could be used to set up a statistical arbitrage strategy on organized option markets. Market makers themselves would find it very useful to blend their intuition about the implied volatility with a normative model, and to have trustworthy estimates of their risks. This would help them to set their bid–ask spread and mid-prices in an optimal way, and to bet on mean reversion. Remember that options are not like stocks: because they have a finite maturity, their price cannot err forever but *has to* converge to the payoff at maturity. Biased prices will therefore sooner or later be corrected.

How could the market be biased and give prices that are too high or too low, leading to statistical arbitrage opportunities? There are several reasons, among them over-reaction, that leads to systematic errors on the correct volatility level, but also sheer supply and demand effects. If out-of-the-money put options on the S&P are considered a very useful insurance to hold, then *some* operators in the market will agree to pay these options above their fair price (again, because the corresponding risk reduction justifies to their eyes the extra cost). One should keep in mind that even 'liquid' option markets are actually not that liquid. In any case, arbitrage can only be statistical and risky: as most other systematic strategies used on financial markets, the probability of being right is never much above 50%, and many sources of error must be carefully dismissed before these strategies can become profitable. This 'high-tech' trading is the speciality of a few hedge funds or prop trading desks (in particular our own company, CFM-Science & Finance [3]).

Trading against the market is fun, but devising reliable option prices becomes mandatory when there is just no market to give us a price. This is the case of all exotic over-the-counter products, or of plain vanilla options on exotic underlyings. For example, we (with Andrew Matacz) recently had to deal with options on hedge funds that CDC-IXIS was writing for its clients. Was their price reasonable? How was CDC-IXIS to calculate and provision its risk? Using Black–Scholes would have been ludicrous. The returns of some hedge funds are tremendously risky, with obese tails. Some of them are only liquid every other month. What is the meaning of the continuous delta-hedge in these circumstances? We developed a specific model, based on a thorough statistical analysis of a large pool of hedge funds, to obtain satisfactory prices and hedges, and reasonable estimates of the risk. More generally, the use of the Black–Scholes formula with an implied volatility and using the Greeks as a rudder is perhaps fine for sailing in a slowly changing environment, but no doubt will leave you stranded whenever a storm sets in. There seems to be plenty of space for improvements.

More refined theories suggest an interesting scenario: the volatility of historical time series is indeed not constant, but the term structure of higher cumulants (the skewness and kurtosis) is relatively stable and, more interesting, quite accurately reflected in the average shape of the market smiles. Jessica James worries that these theories would not be as 'fast and easy to use' as Black–Scholes. This might have been true twenty years ago when computers were lousy and simple analytical formulae vital. But now, computer speed is such that one can implement fully hedged Monte Carlo schemes that allows one to obtain option prices using very complex (but faithful) models in a stroke. An example of a theoretical smile on Microsoft, together with the associated residual risk, is shown in figure 1. This whole curve can be obtained in a few seconds on a standard workstation.

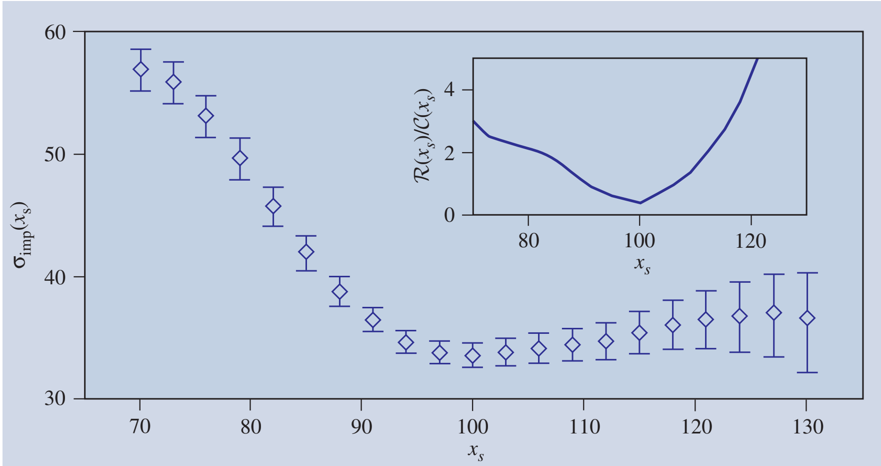

Figure 1. Smile curve obtained using an optimally hedged Monte Carlo with historical data [2], for a one-month option on Microsoft (volatility as a function of strike price x_s). The current price is 100. The error bars are estimated from the residual risk and can be used to set the scale of the bid–ask spread. The inset shows this residual risk $\mathcal{R}$ as a function of strike normalized by the 'time-value' of the option $\mathcal{C}$ (i.e. by the call or put price, whichever is out-of-the-money). Note that the risk is 30% of the option price at the money, and rapidly reaches 100% of the option price as the moneyness grows.

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Further reading

[1] Bouchaud J-P and Potters M 2000 *Theory of Financial Risks* (Cambridge: CUP)
[2] Potters M, Bouchaud J-P and Sestovic D 2001 *Risk* **14** (3) 133
[3] http://www.science-finance.com

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