"Stiff" Field Theory of Interest Rates and Psychological Future Time

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Abstract

The simplest field theory description of the multivariate statistics of forward rate variations over time and maturities, involves a quadratic action containing a gradient squared rigidity term. However, this choice leads to a spurious kink (infinite curvature) of the normalized correlation function for coinciding maturities. Motivated by empirical results, we consider an extended action that contains a squared Laplacian term, which describes the bending stiffness of the FRC. With the extra ingredient of a 'psychological' future time, describing how the perceived time between events depends on the time in the future, our theory accounts extremely well for the phenomenology of interest rate dynamics.

Forward interest rate $f(t,x)$ is the interest rate, agreed upon at time t, for an instantaneous loan to be taken at future time $x > t$, between x and $x + dx$. At any instant of time t, the forward rate curve (FRC) $f(t,x)$ defines a kind of string which moves and deforms with time. Modelling the motion of this curve is of paramount importance for many financial applications [1]: pricing of interest rate derivatives, such as 'caps' that limit the rate of loans that individuals take on their housing, interest swaps, risk management (asset liability management), etc. The industry standard is the so-called HJM model [2, 3]. This model has recently been generalized in different directions [4, 5, 6], in particular by one of us [7, 8, 9], who has proposed a two-dimensional Euclidean quantum field theory for modelling the forward interest rate curve. The forward interest rate dynamics has a drift velocity $\alpha(t,x)$ and volatility $\sigma(t,x)$; it is convenient to define a driftless noise field $A(t,x)$ by

$$\frac{\partial f}{\partial t}(t,x) = \alpha(t,x) + \sigma(t,x)A(t,x). \tag{1}$$

The noise field describes the external shocks of the economy on the different maturities of the forward rates; its statistics is governed by the exponential of an 'action' $S[A]$, which gives the weight of a given path of A in the two dimensional space x, t, and is defined on the semi-infinite domain $x \geq t$. The simplest action that factors in the one dimensional nature of the forward rate string was proposed in [7, 8, 9], and reads

$$S[A] = -\frac{1}{2}\int_{t_0}^{\infty} dt \int_{t}^{\infty} dx \left\{ A^2(t,x) + \frac{1}{\mu^2}\left(\frac{\partial A(t,x)}{\partial x}\right)^2 \right\}, \tag{2}$$

where μ is a 'rigidity' parameter, that gives an elasticity to the FRC. To eliminate boundary terms from the action we choose to impose Neumann boundary condition, i.e.

$$\frac{\partial A(t,x)}{\partial x}\bigg|_{x=t} = 0, \tag{3}$$

corresponding to a parallel motion of the FRC for short maturities, which is reasonable since the spot rate $f(t,t)$ is fixed by the central bank, and very short maturities carry no extra risk.

The quantum field theory [10] needed to describe the statistics of the FRC is defined by a functional integral over all variables $A(t,x)$ and yields the partition function $Z = \int DAe^{S[A]}$. The propagator (or noise correlator) is given by

$$\langle A(t,x)A(t',x')\rangle = \frac{1}{Z}\int DA\ A(t,x)A(t',x')e^{S[A]} \equiv \delta(t-t')D(x,x';t). \tag{4}$$

Since the above action is Gaussian, the market defined by the above model is complete in the sense that all contingent claims can be perfectly replicated by hedging appropriately, as in the standard Black-Scholes model. For consistency of the description, one should further impose a 'martingale' condition that reads [7]

$$\alpha(t,\theta) = \sigma(t,\theta)\int_0^{\theta} d\theta' D(\theta,\theta')\sigma(t,\theta').$$

Note however that this term is usually numerically very small [12]. With the above choice of the action, the propagator, in new co-ordinates $\theta_\pm$ is given by

$$D(\theta_+,\theta_-) = \frac{\mu}{2}[e^{-\mu\theta_+} + e^{-\mu|\theta_-|}] \tag{5}$$

with $\theta_\pm = \theta \pm \theta'$, $\theta = x - t$ and $\theta' = x' - t$ with $x, x' > t$.

Note that the slope of the propagator perpendicular to $\theta_- = 0$, as in Figure 1, is *discontinuous* across the diagonal

$$m = \frac{\partial D(\theta_+;\theta_-)}{\partial \theta_-}\bigg|_{\theta_-=0} = \frac{\mu^2}{2}\begin{cases} -1 & \theta_- > 0 \\ +1 & \theta_- < 0 \end{cases} \tag{6}$$

All the variants of the propagator based on a gradient squared rigidity term in the action show a similar infinite curvature singularity along the diagonal [11].

However, as discussed in [11], the surface of the empirical propagator given in Figure 2, is extremely smooth and shows no such kinks. This observation is in fact related to the

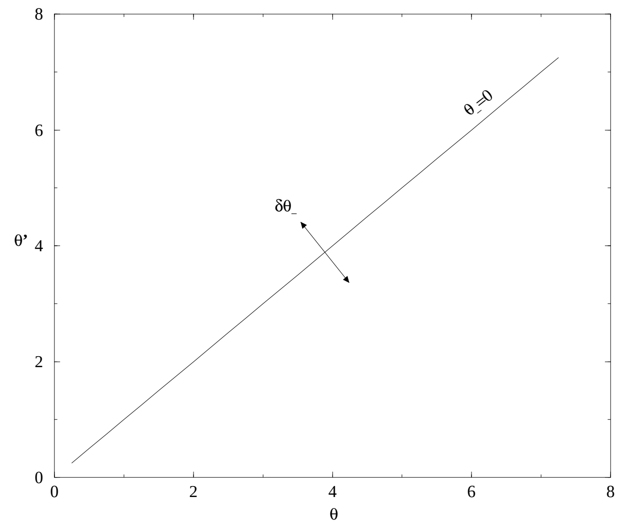

Figure 1: For $\theta_- = \theta - \theta'$, the figure shows that the diagonal axis is given by $\theta_- = 0$. The direction of change in θ_- for constant θ_+, namely $\delta\theta_-$, is orthogonal to the diagonal, as shown in the Figure.

empirical study of [12] (see also [13, 14]), where the correlation of the (fixed maturity) forward rates daily variations $\delta f(t, \theta) \equiv f(t+\epsilon, t+\epsilon+\theta) - f(t, t+\theta)$ was studied. More precisely, the eigenvectors $\Psi_q(\theta)$ and eigenvalues ζ_q, $q = 1, 2, ...$, of the correlation matrix $\mathcal{N}(\theta, \theta')$, defined as

$$\begin{aligned} \mathcal{N}(\theta, \theta') &= < \delta f(t, \theta)\delta f(t, \theta') >_c \\ &\approx \epsilon\sigma(\theta)\sigma(\theta')D(\theta, \theta') \end{aligned}$$

were determined. It was found that the eigenvectors show a structure similar to the modes of a vibrating string (Ψ_1 has no nodes, Ψ_2 has one node, etc. - see also [15]), and that the eigenvalues ζ_q behave as $(a + bq^2)^{-1}$ for small q (where a, b are constants), crossing over to a faster decay $\approx q^{-4}$ for larger q's. It is clear that the square gradient action does indeed lead to plane-wave eigenvectors, with eigenvalues given by $(a+bq^2)^{-1}$ with $b \propto \mu^{-2}$. However, the q^{-4} falloff points to the existence of another term in the action that is a fourth power in the derivative along the future time x. This higher power of the derivative *stiffens* the fluctuations of the forward rates curve, in the sense that two nearby maturities experience more correlated external shocks. The aim of the rest of this letter is to compare in details empirical data with the predictions of the following action

$$S_Q[A] = -\frac{1}{2}\int_{t_0}^{\infty} dt \int_t^{\infty} dx \left\{ A^2(t,x) + \frac{1}{\mu^2}\left(\frac{\partial A(t,x)}{\partial x}\right)^2 + \frac{1}{\lambda^4}\left(\frac{\partial^2 A(t,x)}{\partial x^2}\right)^2 \right\}, \qquad (7)$$

that includes the new stiffness term. We will see that this theory indeed allows one to get rid of the infinite curvature of the propagator along the diagonal, but that a quantitative agreement with empirical data can only be achieved if the action is written in terms of an effective 'psychological' time z in the maturity direction, that is a sublinear function of the 'true' maturity $x - t$. In other words, changes in the maturity direction are given by $\partial/\partial z$. The introduction of z is similar (but not equal [11]) to the rigidity μ^{-2} and the stiffness λ^{-2} not being constant along the x direction, which should be expected. With this extra ingredient, we will see that the details of the propagator surface are reproduced with surprising accuracy.

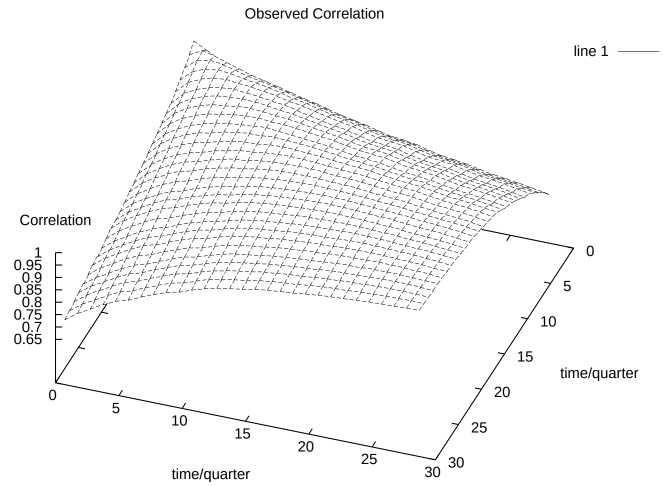

Figure 2: Empirical correlation $\mathcal{C}(\theta,\theta') = \frac{\langle \delta f(t,\theta)\delta f(t,\theta')\rangle_c}{\sqrt{\langle \delta f^2(t,\theta)\rangle_c}\sqrt{\langle \delta f^2(t,\theta')\rangle_c}}$, determined from the Eurodollar rates in the period 1994-1996. See [12] for more details on the data.

Let us first compute the propagator of the 'stiff' action above. Using eq.(5), we find

$$G(\theta_+;\theta_-) = \left(\frac{\lambda^4}{\alpha_+ - \alpha_-}\right)\left[\frac{1}{\alpha_-}D(\theta_+;\theta_-;\sqrt{\alpha_-}) - \frac{1}{\alpha_+}D(\theta_+;\theta_-;\sqrt{\alpha_+})\right] \tag{8}$$

with

$$\alpha_\pm = \frac{\lambda^4}{2\mu^2}\left[1 \pm \sqrt{1 - 4(\frac{\mu}{\lambda})^4}\ \right]$$

In the limit $\lambda \to \infty$ one finds $\alpha_+ \simeq \lambda^4/\mu^2$ and $\alpha_- \simeq \mu^2$. Hence the propagator has the following limit

$$\lim_{\lambda\to\infty} G(\theta_+;\theta_-;\mu,\lambda) \to D(\theta_+;\theta_-;\mu)$$

and, as expected, reduces to the case of constant rigidity.

The solution for $\alpha_\pm$ yields three distinct cases, namely, when $\alpha_\pm$ is real, complex or degenerate depending on whether $\mu < \sqrt{2}\lambda, \mu > \sqrt{2}\lambda, \mu = \sqrt{2}\lambda$ respectively. One finds

$$G(\theta_+;\theta_-) =$$

$$\begin{cases} \frac{\lambda}{2\sinh(2b)}\left[e^{-\lambda\theta_+\cosh(b)}\sinh\{b+\lambda\theta_+\sinh(b)\} + e^{-\lambda|\theta_-|\cosh(b)}\sinh\{b+\lambda|\theta_-|\sinh(b)\}\right] \\ \frac{\lambda}{4}\left[e^{-\lambda\theta_+}\{1+\lambda\theta_+\} + e^{-\lambda|\theta_-|}\{1+\lambda|\theta_-|\}\right] \\ \frac{\lambda}{2\sin(2\phi)}\left[e^{-\lambda\theta_+\cos(\phi)}\sin\{\phi+\lambda\theta_+\sin(\phi)\} + e^{-\lambda|\theta_-|\cos(\phi)}\sin\{\phi+\lambda|\theta_-|\sin(\phi)\}\right] \end{cases} \tag{9}$$

In the above equation, $\alpha_\pm = \lambda^2 e^{\pm b}$ in the first case, and $\alpha_\pm = \lambda^2 e^{\pm i\phi}$ in the third case, and $b = \phi = 0$ in the degenerate case. Expanding the propagator $G(\theta_+;\theta_-)$ about $\theta_- = 0$ leads to a cancellation of the term linear in $|\theta_-|$ and gives a final result that is a function of θ_-^2. More precisely, the curvature r_Q orthogonal to the diagonal line $\theta_- = 0$ is given, in the real case, by

$$r_Q = \frac{\partial^2 G_b(\theta_+;\theta_-)}{\partial\theta_-^2}|_{\theta_-=0} = -\frac{\lambda^3\sinh(b)}{2\sinh(2b)}\left[\cosh^2(b) - \sinh(b)\right] < 0. \tag{10}$$

$r_Q < 0$ follows from the fact that $b \geq 0$, confirming that the value of the propagator along $\theta_- = 0$ is a maximum. A similar result holds in the complex case. Note that in the limit of $\lambda \to \infty$, one can no longer carry out the Taylor expansion around $\theta_- = 0$, the cancellation of the term linear in $|\theta_-|$ becomes invalid, and the propagator $G(\theta_+;\theta_-)$ develops the expected kink.

In order to compare with empirical data, we define the normalized correlation function by

$$\mathcal{C}(\theta,\theta') = \frac{G(\theta,\theta')}{\sqrt{G(\theta,\theta)G(\theta',\theta')}} \tag{11}$$

Of special interest will be the curvature of $\mathcal{C}(\theta,\theta')$ perpendicular to the diagonal, for which an explicit expression can be obtained. We do not show this formula here but note that the curvature is predicted to *increase* with θ_+ (see Fig 3, inset). As one moves along the diagonal to longer maturities, the (negative) curvature of $\mathcal{C}(\theta,\theta')$ increases, which means that the noise affecting nearby maturities is faster to decorrelate as a function of $\theta - \theta'$. This is contrary to intuition: since the long term future is much more uncertain, one feels that shocks in the far future are more difficult to resolve temporally than shocks in the near future. Therefore one expects, and indeed empirically find, that the curvature is a decreasing function of the maturity. In fact, we have discovered the new following 'stylized fact': the curvature R of the FRC correlation function along the diagonal decays as a power law of the maturity, $R(\theta_+) \sim \theta_+^{-\nu}$, with $\nu \approx 1.32$ (see Fig 3, inset).

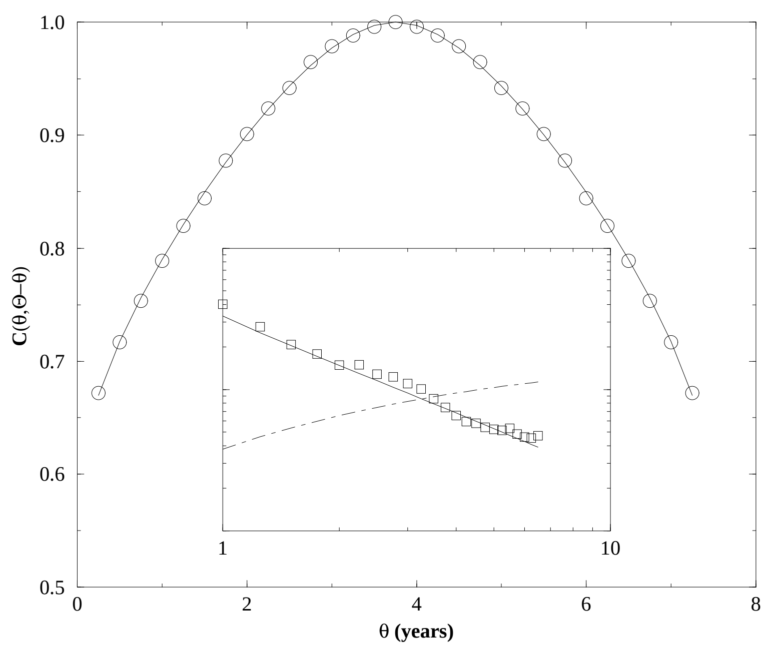

Figure 3: Circles: Empirical correlation $\mathcal{C}(\theta;\theta')$ along the longest stretch perpendicular to the diagonal, i.e. $\theta' = \Theta - \theta$, where Θ is the maximum available maturity. The plain line is the best fit with our stiff propagator model and a power-law psychological time. The inset shows a plot, in log-log coordinates, of the curvature $-R(\theta_+)$ as a function of θ_+, for a) the stiff model in real (physical) time, showing an increasing curvatue (dashed line) and b) the stiff model with a power law psychological time $z(\theta)$, correctly sloping downwards as to reproduce the power law behaviour of the empirical curvature (squares).

A way to reconcile the above finding with our stiff quantum field theory is to realize that the gradient terms in the action need not be uniform. More intuitively, in the mind of market participants, the perceived time between events depends on the time in the future, and a decreasing function of the maturity itself. The distance (in time) between - say - a 10 years maturity and a 30 years maturity is clearly much less than the distance between a 1 month maturity and a 10 year maturity. A way to describe this mathematically is to replace the true, physical future time $\theta = x - t$ by a *psychological* time $z = z(\theta)$ [11, 16], which is expected to grow sublinearly with θ, so that the rate of time change $z'(\theta)$ is a decreasing function of θ. In general, one can impose some general features of function $z(\theta)$: it is monotone increasing, such that $\theta = \theta(z)$ is well defined, and one can impose $z(0) = 0$; $z(\infty) = \infty$. The independent variables will now be $t, z(\theta)$ instead of t, x. Our final model for the forward rate dynamics then reads:

$$\frac{\partial f}{\partial t}(t,\theta) = \alpha(t, z(\theta)) + \sigma(t, z(\theta))A(t, z(\theta)) \quad ; \quad \theta = x - t \tag{12}$$

where $f(t,\theta)$ depends only on calendar time $\theta = x - t$. An important feature of the defining equation above is that both future times, namely $\theta = x - t$ and psychological time $z(\theta)$ occur in the theory [1]. The corresponding stiff action in psychological time is written as

$$S_z = -\frac{1}{2}\int_{t_0}^{\infty} dt \int_0^{\infty} dz \left(A^2 + \frac{1}{\mu^2}\left(\frac{\partial A}{\partial z}\right)^2 + \frac{1}{\lambda^4}\left(\frac{\partial^2 A}{\partial z^2}\right)^2 \right) \tag{13}$$

The propagator for S_z is $G(z, z'; \mu, \lambda)$ as in eq.(8) and the martingale condition for psychological time is given by [7] $\alpha(t, z) = \sigma(t, z) \int_{z(0)}^{z} dz' G(z, z')\sigma(t, z')$. Our final result on the normalized correlation now reads

$$\begin{aligned} \mathcal{C}_{Qz}(\theta+;\theta_-) &= \frac{g_+(z_+) + g_-(z_-)}{\sqrt{[g_+(z_+ + z_-) + g_-(0)][g_+(z_+ - z_-) + g_-(0)]}} \\ z_\pm(\theta_+;\theta_-) &\equiv z(\theta) \pm z(\theta') \end{aligned} \tag{14}$$

with, in the real case that will be of relevance for fitting the empirical data

$$\begin{aligned} g_+(z) &= e^{-\lambda z \cosh(b)} \sinh\{b + \lambda z \sinh(b)\} \\ g_-(z) &= e^{-\lambda |z| \cosh(b)} \sinh\{b + \lambda |z| \sinh(b)\} \\ e^{\pm b} &= \frac{\lambda^2}{2\mu^2}\left[1 \pm \sqrt{1 - 4(\frac{\mu}{\lambda})^4}\ \right] \end{aligned}$$

It can be shown [7] that the curvature with psychological time reads

$$\frac{\partial^2 \mathcal{C}_{Qz}(\theta_+;0)}{\partial \theta_-^2} = [z'(\theta_+)]^2 R_Q(2z(\theta_+/2)), \tag{15}$$

where R_Q is the curvature of the model in physical time. Since one observes a power law fall off for the curvature, we can make the ansatz $z(\theta) = \theta^\eta$ for fitting the data. Using the

[1] The theory for psychological future time can be defined entirely in terms of forward rates $\tilde{f}(t, z(\theta))$. However, for imposing the martingale condition, it is necessary [7] to specify the relation between $\tilde{f}(t, z(\theta))$ and $f(t,\theta)$, and in effect one would recover Eq.(12)

fact that $R_Q(2z(\theta_+/2))$ varies very slowly as a function of θ_+, one can make the following approximation

$$[z'(\theta_+)]^2 \propto \frac{1}{\theta_+^\nu} \Rightarrow 2\eta - 2 = -\nu, \tag{16}$$

leading to $\eta \approx 0.34$. Therefore the psychological time flows, as expected, much slower than real time. The rate of change of psychological time decreases as $\approx \theta^{-0.66}$: a year after ten years looks similar to two weeks after a month.

Having used the behaviour of the curvature to fix the value of η (and thus, up to an irrelevant overall scale, the function $z(\theta)$, we are left with only two parameters, λ and μ, to fit the whole correlation surface $\mathcal{C}(\theta, \theta')$. For the Eurodollar data that we have used (see [12] for more details), we have up to 30 different maturities and therefore 405 different points (the diagonal values are trivial). We determine λ and μ such as to minimize the average square error between the empirical $\mathcal{C}(\theta, \theta')$ and the prediction of the model. Defining $\tilde{\lambda}^\eta = \lambda$ and $\tilde{\mu}^\eta = \mu$, the best fit is obtained for $\tilde{\lambda} = 1.79$/year and $\tilde{\mu} = 0.403$/year, corresponding to $b = 0.845$. The residual error is as low as 0.4% per point, and the order of magnitude of the time scales (one year) defined by $\tilde{\lambda}$ and $\tilde{\mu}$ are very reasonable. The remarkable quality of the fit can be checked in Fig.3, along the longest stretch perpendicular to the diagonal, i.e. $\theta' = \Theta - \theta$, where Θ is the maximum available maturity (7.5 years). Even more remarkable is that the curvature of $\mathcal{C}(\theta, \theta')$ along the diagonal is very precisely reproduced by the same fit, as shown in the inset of Fig. 3. Testing the fit on the second derivative of the fitted surface is of course much more demanding. The existence of the boundary at $x = t$, or $\theta = 0$, is reflected in the θ_+ term in the propagator; if one removes this term, and in effect assumes that the forward rates exist for all $-\infty \leq x \leq +\infty$, then the fit deteriorates with the root mean square error increasing from 0.40% to 0.53%. The existence of the boundary at $x = t$ can hence be seen to have a significant effect on the correlation of the forward rates.

Let us summarize what we have achieved in this study. The simplest field theoretical description of the multivariate statistics of forward rate variations over time t and maturities θ, involves a quadratic action containing a gradient squared rigidity term [8], that captures the one dimensional string nature of the forward rate curve. However, this choice leads to a spurious kink (infinite curvature) of the normalized correlation function along the diagonal $\theta = \theta'$. Motivated by a previous empirical study [12] where the short wavelength fluctuations of the FRC were shown to be strongly reduced as compared to that of an elastic string, we have considered an extended action that contains a squared Laplacian term, which describes the bending stiffness of the FRC. In this formulation, the infinite curvature singularity is rounded off. In order to fit to the observed correlation functions of the forward interest rates, however, one has to add as an extra ingredient that the rigidity/stiffness constants are in fact not constant along the maturity axis but increase with maturity. An intuitive and parsimonious way to describe this effect is to postulate that markets participants, who generate the random evolution of the FRC, do not perceive future time in a uniform manner. Rather, time intervals in the long term future are shrunk. The introduction of a 'psychological time' $z(\theta)$, found to be a power law of the true time, allows one to provide an excellent fit of empirical data, and in particular to reproduce accurately a new stylized fact: the curvature of the forward rate correlation perpendicular to the diagonal decays as a power-law of the maturity. We believe that our quantum field formulation, including the new stiffness term and coupled with an appropriate calibration of the term structure of the volatility $\sigma(\theta)$ (see [12, 13]), accounts extremely well for the phenomenology of interest rate dynamics. It is also mathematically tractable, and should allow one to compute in closed forms derivative prices and optimal hedging strategies. It would be interesting to generalize the above model to account for non Gaussian effects, that are important in many cases [14]. This would amount to considering non quadratic terms in A in the action. We leave this extension for future work.

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