arXiv:1603.04364v1 [cond-mat.stat-mech] 14 Mar 2016

On the overlaps between eigenvectors of correlated random matrices

Joël Bun[1,2,3], Jean-Philippe Bouchaud[1], Marc Potters[1]
[1] *Capital Fund Management, 23–25, rue de l'Université, 75 007 Paris*
[2] *LPTMS, CNRS, Univ. Paris-Sud, Université Paris-Saclay, 91405 Orsay, France and*
[3] *Leonard de Vinci Pôle Universitaire, Finance Lab, 92916 Paris La Défense, France.*

We obtain general, exact formulas for the overlaps between the eigenvectors of large correlated random matrices, with additive or multiplicative noise. These results have potential applications in many different contexts, from quantum thermalisation to high dimensional statistics. We apply our results to the case of empirical correlation matrices, that allow us to estimate reliably the width of the spectrum of the "true" underlying correlation matrix, even when the latter is very close to the identity matrix. We illustrate our results on the example of stock returns correlations, that clearly reveal a non trivial structure for the bulk eigenvalues.

The structure of the eigenvalues and eigenvectors of large random matrices is of primary importance in many different contexts, from quantum mechanics to high dimensional data analysis. Correspondingly, Random Matrix Theory (RMT) has established itself as a major discipline, at the frontier between theoretical physics, mathematics, probability theory, and applied statistics, with a somewhat intimidating corpus of knowledge [1]. One of the most striking applications of RMT concerns quantum chaos and quantum transport [2], with renewed interest coming from problems of quantum ergodicity ("eigenstate thermalisation") [3, 4], entanglement and dissipation (for recent reviews see [5, 6]). In the context of signal processing, RMT is of primary importance in the analysis of high dimensional statistics [7–9], wireless communication channels [10, 11], etc. Other examples cover the dynamics of complex systems – from random ecologies [12] to glasses and spin-glasses [13].

Whereas the spectral properties of random matrices have been investigated at length, the interest has recently shifted to the statistical properties of their eigenvectors – see e.g. [3, 14] and [15–21] for more recent papers. In particular, one can ask how the eigenvectors of a sample matrix $\mathbf{S}$ resemble those of the *population* (or *pure*) matrix $\mathbf{C}$ itself. We recently obtained in [22] explicit formulas for the overlaps between these pure and noisy eigenvectors for a wide class of random matrices, generalizing results obtained for sample covariance matrices of [15] – obtained as $\mathbf{S} = \sqrt{\mathbf{C}}\boldsymbol{\mathcal{W}}\sqrt{\mathbf{C}}$ where $\boldsymbol{\mathcal{W}}$ is a Wishart matrix – and for matrices of the form $\mathbf{S} = \mathbf{C} + \mathbf{W}$, where $\mathbf{W}$ is a Gaussian random matrix [18, 23, 24]. In the present paper, we want to generalize these results to the overlaps between the eigenvectors of *two* different realisations of such random matrices, that remain correlated through their common part $\mathbf{C}$. For example, imagine one measures the sample covariance matrix of the same process, but on two non-overlapping time intervals, characterized by two independent realizations of the Wishart noises $\boldsymbol{\mathcal{W}}$ and $\boldsymbol{\mathcal{W}}'$. How close are the corresponding eigenvectors expected to be? We provide exact, explicit formulas for these overlaps in the high dimensional regime. Perhaps surprisingly, these overlaps can be evaluated from the empirical spectrum of $\mathbf{S}$ only, i.e. *without any prior knowledge* of the pure matrix $\mathbf{C}$ itself. This leads us to propose a statistical test based on these overlaps, that allows one to determine whether two realisations of the random matrix $\mathbf{S}$ and $\mathbf{S}'$ indeed correspond to the very same underlying "true" matrix $\mathbf{C}$ either in the multiplicative and additive cases defined above. We give a transparent interpretation to our formulas and generalize them to various cases, in particular when the noises are correlated.

Throughout the following, we consider $N \times N$ symmetric random matrices and denote by $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_N$ the eigenvalues of $\mathbf{S}$ and by $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N$ the corresponding eigenvectors. Similarly, we denote by $\lambda'_1 \geqslant \lambda'_2 \geqslant \ldots \geqslant \lambda'_N$ the eigenvalues of $\mathbf{S}'$ and by $\mathbf{u}'_1, \mathbf{u}'_2, \ldots, \mathbf{u}'_N$ the associated eigenvectors. Note that we will index the eigenvectors by their corresponding eigenvalues for convenience. The central object that we focus on in this study are the asymptotic ($N \to \infty$) *scaled, mean squared overlaps*

$$\Phi(\lambda, \lambda') := N\mathbb{E}(\mathbf{u}_\lambda \cdot \mathbf{u}'_{\lambda'})^2, \qquad (1)$$

that remain $O(1)$ in the limit $N \to \infty$. In the above equation, the expectation $\mathbb{E}$ can be interpreted either as an average over different realisations of the randomness or, for fixed randomness, as an average over small "slices" of eigenvalues of width $\eta = \mathrm{d}\lambda \gg N^{-1}$, such that the result becomes self-averaging in the large N limit. We will study the asymptotic behavior of (1) using the complex function

$$\psi(z, z') := \left\langle \frac{1}{N} \operatorname{Tr}\left[(z - \mathbf{S})^{-1}(z' - \mathbf{S}')^{-1}\right] \right\rangle_{\mathcal{P}}, \qquad (2)$$

where $z, z' \in \mathbb{C}$ and $\langle\cdot\rangle_{\mathcal{P}}$ denotes the average with respect to probability measure associated to $\mathbf{S}$ and $\mathbf{S}'$. Indeed, using the spectral decomposition, we obtain the following inversion formula, valid when $N \to \infty$ for any $\lambda, \lambda' \in \operatorname{supp} \varrho$, where ϱ is the spectral density of $\mathbf{S}$ (that we assume here to be the same as that of $\mathbf{S}'$, but see

Appendix for more general cases):

$$\Phi(\lambda,\lambda') = \frac{\mathrm{Re}[\psi_0(z,\overline{z'}) - \psi_0(z,z')]}{2\pi^2 \varrho(\lambda)\varrho(\lambda')}, \tag{3}$$

with $z = \lambda - \mathrm{i}\eta$, $\overline{z}$ its complex conjugate and $\psi_0 \equiv \lim_{\eta\downarrow 0}\psi$. The derivation of the identity (3) can be found in the Appendix.

The study of the asymptotic behavior of the function ψ requires to control the resolvent of $\mathbf{S}$ and $\mathbf{S}'$ entry-wise. It was shown recently that one can approximate these (random) resolvents by deterministic equivalent matrices [22, 25, 26]. In the case of correlated Wishart (covariance) matrices $\mathbf{S} = \sqrt{\mathbf{C}}\boldsymbol{\mathcal{W}}\sqrt{\mathbf{C}}$, this allows us to obtain, for $N \to \infty$:

$$\psi(z,z') = \frac{\zeta(z')g(z) - \zeta(z)g(z')}{z'\zeta(z') - z\zeta(z)}, \tag{4}$$

where $g(z)$ is the Stieltjes transform of $\mathbf{S}$: $g(z) = N^{-1}\,\mathrm{Tr}(zI_N - \mathbf{S})^{-1}$, and $\zeta(z) := (1-q+qzg(z))^{-1}$, where $q = N/T$ is e.g. the ratio between the number of observables and the length of the empirical time series. The derivation of (4) is given in the Appendix. In the case of *additive* Gaussian noise, $\mathbf{S} = \mathbf{C}+\mathbf{W}$, one obtains instead:

$$\psi_a(z,z') = \frac{g(z) - g(z')}{\xi(z') - \xi(z)}, \tag{5}$$

where $\xi(z) = z - \sigma^2 g(z)$ and σ^2 is the variance of the elements of the Gaussian random matrix $\mathbf{W}$. [Here and henceforth, the subscript a denotes the additive noise.] Note that from Ref. [26] we expect the results (4) and (5) to hold with fluctuations of order $N^{-1/2}$.

One notices that both Eq. (4) and Eq. (5) only depend on *a priori* observable quantities, i.e. they do not involve explicitly the unknown matrix $\mathbf{C}$. Hence, we will obtain an observable expression for (1) using the inversion formula (3). For sample covariance matrices, it is convenient to work with $m(z) := (z\zeta(z))^{-1}$. Defining $m_0(\lambda) = \lim_{\eta\downarrow 0} m(\lambda - \mathrm{i}\eta) \equiv m_R(\lambda) + \mathrm{i}m_I(\lambda)$, the asymptotic behaviour of Eq. (1) for any $\lambda, \lambda' \in \mathrm{supp}\,\varrho$ can be explicitly computed as (see Appendix for details)

$$\Phi(\lambda,\lambda') \underset{N\to\infty}{\sim} 2q\frac{\alpha(\lambda,\lambda')}{\lambda\,\lambda'\,\gamma(\lambda,\lambda')}, \tag{6}$$

where α and γ are deterministic and symmetric functions

$$\begin{aligned}\alpha(\lambda,\lambda') &:= (\lambda-\lambda')\big(m_R(\lambda)|m_0(\lambda')|^2 - m_R(\lambda')|m_0(\lambda)|^2\big)\\ \gamma(\lambda,\lambda') &:= \big[(m_R(\lambda)-m_R(\lambda'))^2 + (m_I(\lambda)+m_I(\lambda'))^2\big]\times\\ &\quad\;\big[(m_R(\lambda)-m_R(\lambda'))^2 + (m_I(\lambda)-m_I(\lambda'))^2\big]\end{aligned} \tag{7}$$

The result (6) is indeed remarkable in that the mean squared overlap between the eigenvectors of independent sample covariance matrices can be expressed to leading order in N without the explicit knowledge of the true underlying covariance $\mathbf{C}$. Similarly, in the additive case, we find:

$$\Phi_a(\lambda,\lambda') \underset{N\to\infty}{\sim} 2\sigma^2(\lambda-\lambda')\frac{\xi_R(\lambda)-\xi_R(\lambda')}{\gamma_a(\lambda,\lambda')}, \tag{8}$$

where $\gamma_a(\lambda,\lambda')$ is given by the same expression as $\gamma(\lambda,\lambda')$ in Eq. (7) with the substitutions $m_R \to \xi_R$ and $m_I \to \xi_I$. Again, (8) does not involve the pure matrix $\mathbf{C}$.

Eqs. (6) and (8) are exact and are the main new results of this work. They can be simplified in the limit where $\lambda' \to \lambda$, i.e. when one studies the "self" overlaps between eigenvectors corresponding to the same eigenvalue λ. In this case, we find:

$$\Phi(\lambda,\lambda) = \frac{q}{2\lambda^2}\frac{|m_0(\lambda)|^4\partial_\lambda\big[m_R(\lambda)/|m_0(\lambda)|^2\big]}{m_I^2(\lambda)|\partial_\lambda m_0(\lambda)|^2}, \tag{9}$$

and for the additive case:

$$\Phi_a(\lambda,\lambda) = \frac{\sigma^2}{2}\frac{\partial_\lambda\xi_R(\lambda)}{\xi_I^2(\lambda)\big|\partial_\lambda\xi_0(\lambda)\big|^2}, \tag{10}$$

where we used the notation $\xi_0(\lambda) \equiv \lim_{\eta\downarrow 0}\xi(\lambda - \mathrm{i}\eta)$.

Before investigating some concrete applications of these formulas, let us give an interesting interpretation of the above formalism. We first introduce the set of eigenvectors $\mathbf{v}_\mu$ of the pure matrix $\mathbf{C}$, labelled by the eigenvalue μ. We define the overlaps between the $\mathbf{u}$'s and the $\mathbf{v}$'s as $\sqrt{O(\mu,\lambda)/N}\times\varepsilon(\mu,\lambda)$, where $O(\mu,\lambda)$ were explicitly computed in [22] for a wide class of problems, and $\varepsilon(\mu,\lambda)$ are random variables of unit variance. Now, one can always decompose the $\mathbf{u}$'s as:

$$\mathbf{u}_\lambda = \frac{1}{\sqrt{N}}\int \mathrm{d}\mu\,\varrho_C(\mu)\sqrt{O(\mu,\lambda)}\varepsilon(\mu,\lambda)\mathbf{v}_\mu, \tag{11}$$

where ϱ_C is the spectral density of $\mathbf{C}$. Using the orthonormality of the $\mathbf{v}$'s, one then finds:

$$\mathbf{u}_\lambda\cdot\mathbf{u}'_{\lambda'} = \frac{1}{N}\int \mathrm{d}\mu\,\varrho_C(\mu)\sqrt{O(\mu,\lambda)O(\mu,\lambda')}\,\varepsilon(\mu,\lambda)\varepsilon(\mu,\lambda'). \tag{12}$$

If we square this last expression and average over the noise, and make an "ergodic hypothesis" [3] according to which all signs $\varepsilon(\mu,\lambda)$ are in fact independent from one another, one finds the following, rather intuitive convolution result for square overlaps:

$$\Phi(\lambda,\lambda') = \int \mathrm{d}\mu\,\varrho_C(\mu)O(\mu,\lambda)\,O(\mu,\lambda'). \tag{13}$$

It turns out that this expression is completely general and exactly equivalent to Eqs. (6) and (8) in the corresponding cases. However, whereas this expression still contains some explicit dependence on the structure of the pure matrix $\mathbf{C}$, it has disappeared in Eqs. (6) and (8).

Now, let us consider the case of sample correlation matrices, first in the theoretical case where the true correlation matrix $\mathbf{C}$ is an inverse Wishart matrix of parameter

$\kappa \in (0, \infty)$, that corresponds to $1/q$ for Wishart matrices (see [22] for details). In this case, the function $m(z)$ can be explicitly computed. This finally leads to:

$$\Phi(\lambda, \lambda) = \frac{\nu(\lambda + 2q\kappa)^2}{2q\kappa\big(2\lambda(\nu + \kappa) - \lambda^2\kappa + \kappa(2q\nu - 1)\big)} \qquad (14)$$

with $\nu := 1 + q\kappa$ and λ is within the interval $[\lambda^-, \lambda^+]$, where the edges are given by $\lambda^\pm = \kappa^{-1}\Big[\nu + \kappa \pm \sqrt{(2\kappa+1)(2q\kappa+1)}\Big]$. An interesting limit corresponds to $\kappa \to \infty$, where $\mathbf{C}$ tends to the identity matrix, and the overlaps are expected to become all equal to $1/N$. Indeed one finds, for a fixed q:

$$\Phi(\lambda, \lambda') \underset{\kappa\to\infty}{\sim} \left[1 + \frac{(\lambda - 1)(\lambda' - 1)}{2q^2}\kappa^{-1} + O(\kappa^{-2})\right], \qquad (15)$$

which is in fact *universal* in this limit, provided the eigenvalue spectrum of $\mathbf{C}$ has a variance given by $(2\kappa)^{-1} \to 0$. [29] This formula is interesting insofar as it allows one to estimate the width of the eigenvalue distribution of $\mathbf{C}$, even when it is close to the identity matrix, i.e. $\kappa \gg 1$. One could think of directly using information on the empirical spectrum, for example the Marčenko-Pastur prediction $\operatorname{Tr}\mathbf{C}^{-1} = (1-q)\operatorname{Tr}\mathbf{S}^{-1}$, that in principle allows one extract the parameter κ through $1 + (2\kappa)^{-1} = (1-q)\operatorname{Tr}\mathbf{S}^{-1}/N$. However, this method is numerically unstable and very imprecise when $\kappa \gg 1$ and finite Ns (for one thing, the RHS can be negative, leading to a negative variance). Our formula based on overlaps avoid these difficulties. As an illustration, we check the validity of Eq. (14) in the inset of Figure 1 in the case $\kappa = 10$, $N = 500$ and $q = 0.5$. We determine the empirical average overlap as follows: we consider 100 independent realisation of the Wishart noise $\boldsymbol{\mathcal{W}}$. For each pair of samples we compute a smoothed overlap as:

$$[(\mathbf{u}_i \cdot \mathbf{u}'_i)^2] = \frac{1}{Z_i}\sum_{j=1}^{N} \frac{(\mathbf{u}_i \cdot \mathbf{u}'_j)^2}{(\lambda_i - \lambda'_j)^2 + \eta^2}, \qquad (16)$$

with $Z_i = \sum_{k=1}^{N}((\lambda_i - \lambda'_k)^2 + \eta^2)^{-1}$ the normalization constant and η the width of the Cauchy kernel, that we choose to be $N^{-1/2}$ in such a way that $N^{-1} \ll \eta \ll 1$. We then average this quantity over all pairs for a given value of i to obtain $[(\mathbf{u}_i \cdot \mathbf{u}'_i)^2]_e$, and plot the resulting quantity as a function of the average eigenvalue position $[\lambda_i]_e$, see Figure 1, inset. The agreement with Eq. (14) is excellent, even when the true underlying matrix $\mathbf{C}$ is close to the identity matrix

We now investigate an application to real data, in the case of stock markets and using a bootstrap technique to generate different samples. Specifically, we consider the standardized daily returns of the 450 most liquid US stocks from 2005 to 2012. We randomly split the total period of 1800 days into two non-overlapping subsets of size $T = 900$. We then measure the overlap between the eigenvectors of the empirical correlation matrices $\mathbf{S}$ and $\mathbf{S}'$ corresponding to the two subsets exactly as in Eq. (16) above and average over 100 different bootstrap samples. The results are reported in Figure 1 where we only focus on "bulk" eigenvalues (the top eigenvectors are governed by non trivial dynamics, see e.g. [16]). We conclude from Figure 1 that bulk eigenvalues have a non trivial structure since the mean squared overlaps strongly depart from the null hypothesis and in fact cannot be estimated using the universal formula Eq. (15) as a parabolic fit is clearly unwarranted. A fit with the full inverse Wishart formula (14) leads to $\kappa \approx 0.7$. [30]. A much better fit could be achieved by choosing another prior for the spectrum of $\mathbf{C}$, for example a power-law as in [8]; we however leave this issue for later investigations. The conclusion of this study is that the cross-sample eigenvector overlaps provide precious insights about the structure of the true eigenvalue spectrum, much beyond the information contained in the empirical spectrum itself.

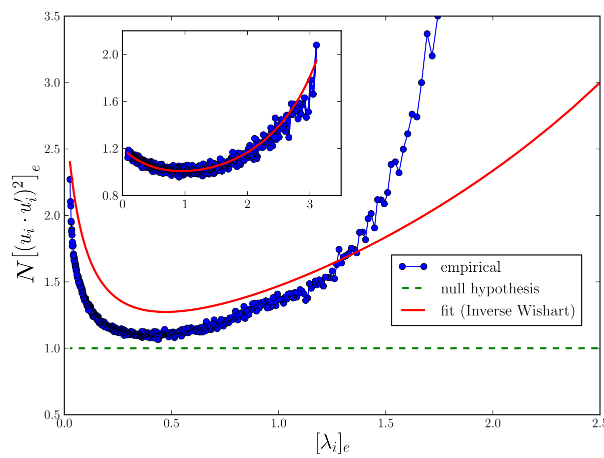

FIG. 1: Main Figure: Evaluation of the self-overlap $N\mathbb{E}(\mathbf{u}_i \cdot \mathbf{u}'_i)^2$ using $N = 450$ US stocks' daily returns from 2005 to 2012 and 100 bootstrap replicas of two non-overlapping periods of size $T = 900$ and with $N = 450$. We compare the empirical results with (i) the null hypothesis $1/N$ (green dotted line) and (ii) an Inverse-Wishart prior with $q = 0.5$ and an estimated parameter $\kappa \approx 0.7$ (red line). Inset: Same plot using synthetic data coming from an Inverse-Wishart prior with parameter $\kappa = 10$ and $N = 500$. The empirical average is taken over 100 realizations of the noise. The agreement with the theoretical formula is excellent.

All the above results can actually be applied in an much broader context (see Appendix for technical details). For instance, the noise parameters can be different for the two realisations: for example the observation ratio $q = N/T$ can be different for $\mathbf{S}$ and $\mathbf{S}'$, or the amplitude of the noise σ can be different in the additive case. Another direction is to consider generalized empirical correlation matrices of the form $\mathbf{S} = \sqrt{\mathbf{C}}\mathbf{OBO}^*\sqrt{\mathbf{C}}$, where $\mathbf{B}$ is a given matrix independent from $\mathbf{C}$, and $\mathbf{O}$ is a random matrix chosen in the Orthogonal group $O(N)$ according to the Haar measure (see e.g. [20, 22, 27]). The case above corresponds to the case where $\mathbf{OBO}^*$ is a Wishart matrix. We find for this general model that (4) still holds with $\zeta(z)$ replaced by $\mathcal{S}_{\mathbf{B}}(zg(z)-1)$ where $\mathcal{S}_{\mathbf{B}}$ is the so-called Voiculescu's S-transform of the $\mathbf{B}$ matrix [28]. If $\mathbf{B}=\boldsymbol{\mathcal{W}}$, then $\mathcal{S}_{\mathbf{B}}(z)=1/(1+qz)$. Similarly, the additive model can be generalized to $\mathbf{S}=\mathbf{C}+\mathbf{OBO}^*$ with the same definitions for $\mathbf{B}$ and $\mathbf{O}$. In that case, the above result (5) again holds with $\xi(z)=z-\mathcal{R}_{\mathbf{B}}(g(z))$, where $\mathcal{R}_{\mathbf{B}}(z)$ is now the $\mathcal{R}$-transform of the $\mathbf{B}$ matrix [28] – which is simply equal to $\mathcal{R}_{\mathbf{B}}(z)=\sigma^2 z$ when $\mathbf{B}=\mathbf{W}$ is a Gaussian random matrix, as considered above. Note that in all these generalized cases, the overlap convolution formula (13) is always valid provided that the noises are independent.

Finally, an important case is when the noises $\mathbf{W}$, $\mathbf{W}'$ or $\boldsymbol{\mathcal{W}}$, $\boldsymbol{\mathcal{W}}'$ are correlated – while the above calculations referred to independent noises. In the additive case, the trick is to realize that one can always write (in law) $\mathbf{W}=\sqrt{\rho}\,\mathbf{W}_0+\sqrt{1-\rho}\,\mathbf{W}_1$ and $\mathbf{W}'=\sqrt{\rho}\,\mathbf{W}_0+\sqrt{1-\rho}\,\mathbf{W}_2$, where $\mathbf{W}_1$, $\mathbf{W}_2$ are now independent, as above. Since our formulas do not rely on the common matrix $\mathbf{C}$ that can therefore be replaced by $\mathbf{C}+\sqrt{\rho}\mathbf{W}_0$, (10) trivially holds with σ^2 simply multiplied by $1-\rho$. The corresponding shape of $\Phi_a(\lambda,\lambda)$ for different values of ρ is shown in Fig. 2. We also provide in the inset a comparison with synthetic data for a fixed $\rho=0.54$, $\sigma^2=1$. The empirical average is taken over 200 realizations of the noise and again, the agreement is excellent. The multiplicative model is also interesting since it describes the case of correlation matrices measured on overlapping periods, such that $\mathbf{S}=\sqrt{\mathbf{C}}\,[\boldsymbol{\mathcal{W}}_0+\boldsymbol{\mathcal{W}}]\,\sqrt{\mathbf{C}}$ and $\mathbf{S}'=\sqrt{\mathbf{C}}\,[\boldsymbol{\mathcal{W}}_0+\boldsymbol{\mathcal{W}}']\,\sqrt{\mathbf{C}}$. This case turns out to be more subtle and is the subject of ongoing investigations.

In summary, we have provided general, exact formulas for the overlaps between the eigenvectors of large correlated random matrices, with additive or multiplicative noise. Remarkably, these results do not require the knowledge of the underlying "pure" matrix and have a broad range of applications in different contexts. We showed that the cross-sample eigenvector overlaps provide unprecedented information about the structure of the true eigenvalue spectrum, much beyond that contained in the empirical spectrum itself. For example, the width of the bulk of the spectrum of the true underlying correlation matrix can be reliably estimated, even when the latter is very close to the identity matrix. We have illustrated our results on the example of stock returns correlations, that clearly reveal a non trivial structure for the bulk eigenvalues.

Acknowledgments: We want to thank R. Allez, R. Bénichou, R. Chicheportiche, B. Collins, A. Rej, E. Sérié and D. Ullmo for very useful discussions.

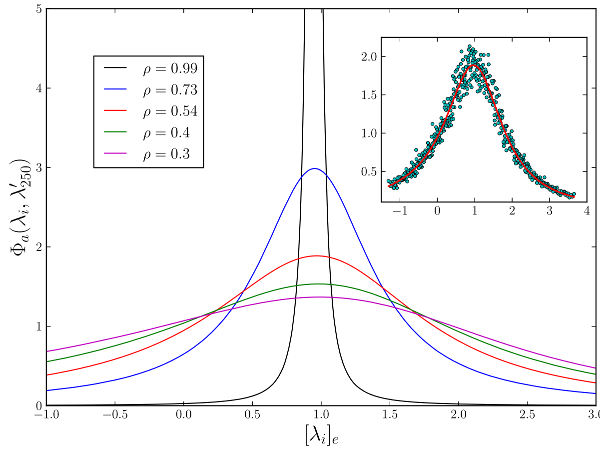

FIG. 2: Main Figure: Evaluation of the self overlap $\Phi_a(\lambda,\lambda')$ for a fixed $\lambda'\approx 0.95$ as a function of λ for $N=500$, $\sigma^2=1$, and for different values of ρ. The population matrix $\mathbf{C}$ is given by a (white) Wishart matrix with parameter $T=2N$. Inset: We compare the theoretical prediction $\Phi_a(\lambda,\lambda'\approx 0.95)$ for a fixed $\rho=0.54$ with synthetic data. The empirical averages (blue points) are obtained from 200 independent realizations of $\mathbf{S}$.

[1] G. Akemann, J. Baik, and P. Di Francesco, *The Oxford handbook of random matrix theory* (Oxford University Press, 2011).
[2] C. W. Beenakker, Reviews of modern physics **69**, 731 (1997).
[3] J. M. Deutsch, Physical Review A **43**, 2046 (1991).
[4] G. Ithier and F. Benaych-Georges, arXiv preprint arXiv:1510.04352 (2015).
[5] R. Nandkishore and D. A. Huse, Annual Review of Condensed Matter Physics **6**, 15 (2015).
[6] J. Eisert, M. Friesdorf, and C. Gogolin, Nature Physics **11**, 124 (2015).
[7] I. M. Johnstone, in *Proceedings on the International Congress of Mathematicians I* (2007), pp. 307–333.
[8] J.-P. Bouchaud and M. Potters, in *The Oxford handbook of Random Matrix Theory* (Oxford University Press, 2011).
[9] D. Paul and A. Aue, Journal of Statistical Planning and Inference **150**, 1 (2014).
[10] A. M. Tulino and S. Verdú, *Random matrix theory and wireless communications*, vol. 1 (Now Publishers Inc, 2004).
[11] R. Couillet, M. Debbah, et al., *Random matrix methods for wireless communications* (Cambridge University Press Cambridge, MA, 2011).
[12] R. M. May, Nature **238**, 413 (1972).
[13] Y. V. Fyodorov, Physical review letters **92**, 240601 (2004).
[14] M. Wilkinson and P. N. Walker, Journal of Physics A: Mathematical and General **28**, 6143 (1995).
[15] O. Ledoit and S. Péché, Probability Theory and Related Fields **151**, 233 (2011).
[16] R. Allez and J.-P. Bouchaud, Physical Review E **86**,

046202 (2012).
[17] P. Bourgade, L. Erdős, H.-T. Yau, and J. Yin, Communications on Pure and Applied Mathematics (2015).
[18] R. Allez and J.-P. Bouchaud, Random Matrices: Theory and Applications **3**, 1450010 (2014).
[19] A. Bloemendal, A. Knowles, H.-T. Yau, and J. Yin, Probability Theory and Related Fields pp. 1–94 (2015).
[20] R. Couillet and F. Benaych-Georges, arXiv preprint arXiv:1510.03547 (2015).
[21] R. Monasson and D. Villamaina, EPL (Europhysics Letters) **112**, 50001 (2015).
[22] J. Bun, R. Allez, J.-P. Bouchaud, and M. Potters, arXiv preprint arXiv:1502.06736 (2015).
[23] Philippe Biane, Quantum Probability Communications **11**, 55 (2003).
[24] R. Allez, J. Bun, and J.-P. Bouchaud, arXiv preprint arXiv:1412.7108 (2014).
[25] Z. Burda, A. Görlich, A. Jarosz, and J. Jurkiewicz, Physica A: Statistical Mechanics and its Applications **343**, 295 (2004).
[26] A. Knowles and J. Yin, arXiv preprint arXiv:1410.3516 (2014).
[27] Z. Burda, J. Jurkiewicz, and B. Wacław, Physical Review E **71**, 026111 (2005).
[28] D. Voiculescu, Inventiones mathematicae **104**, 201 (1991).
[29] The analysis for the additive case leads to a very similar result. More precisely, taking $\mathbf{C} = I_N + \mathbf{W}_0$ with $\mathbf{W}_0$ a GOE (independent from $\mathbf{W}$ and $\mathbf{W}'$) of variance $\sigma_0^2 \to 0$, one finds that $\Phi_a(\lambda, \lambda')$ is given by exactly the same formula (15) with the substitution $2q^2\kappa \to \sigma^4/\sigma_0^2$.
[30] The calibration of κ is performed by least squares using Eq. (14) and where we removed the 10 largest eigenvalues.

Appendix: Supplementary Material

We collect the derivations of the different results in the following sections. For the sake of clarity, we rename $\mathbf{S}'$, its eigenvalues and eigenvectors respectively by $\widetilde{\mathbf{S}}$, $[\widetilde{\lambda}_i]_{i\in[\![1,N]\!]}$ and $[\widetilde{\mathbf{u}}_i]_{i\in[\![1,N]\!]}$, not to confuse "primes" with derivatives. Note that our calculations rely on some physical arguments that can certainly be made rigorous in a mathematical sense. Precise estimates about the error terms for sample covariance matrices or deformed GOE may be obtained using the results of [26] for instance. Throughout the following, the symbol "$\sim$" denotes the limit $N \to \infty$.

Inversion formula (derivation of (3))

The derivation of the inversion formula (3) is pretty straightforward. We start from Eq. (2) that we rewrite using the spectral decomposition of $\mathbf{S}$ and $\widetilde{\mathbf{S}}$ as

$$\psi(z, \widetilde{z}) \;=\; \left\langle \frac{1}{N} \sum_{i,j=1}^{N} \frac{1}{z-\lambda_i} \frac{1}{\widetilde{z}-\widetilde{\lambda}_j} (\mathbf{u}_i \cdot \widetilde{\mathbf{u}}_j)^2 \right\rangle_{\mathcal{P}}, \tag{17}$$

where $\mathcal{P}$ denotes the probability density function of the noise part of $\mathbf{S}$ and $\widetilde{\mathbf{S}}$. For large random matrices, we expect the eigenvalues of $[\lambda_i]_{i\in[\![1,N]\!]}$ and $[\widetilde{\lambda}_i]_{i\in[\![1,N]\!]}$ stick to their *classical* locations, i.e. smoothly allocated with respect to the quantile of the spectral density. Roughly speaking, the sample eigenvalues become deterministic in the large N limit. Hence, we obtain after taking the continuous limit

$$\psi(z, \widetilde{z}) \sim \int\!\!\int \frac{\varrho(\lambda)}{z-\lambda} \frac{\widetilde{\varrho}(\widetilde{\lambda})}{\widetilde{z}-\widetilde{\lambda}} \Phi(\lambda, \widetilde{\lambda}) \mathrm{d}\lambda \mathrm{d}\widetilde{\lambda}, \tag{18}$$

where ϱ and $\widetilde{\varrho}$ are respectively the spectral density of $\mathbf{S}$ and $\widetilde{\mathbf{S}}$, and Φ denotes the mean squared overlap defined in (1) above. Then, it suffices to compute

$$\begin{aligned} \psi(x-\mathrm{i}\eta, y\pm\mathrm{i}\eta) \;&\sim\; \int\!\!\int \frac{(x-\lambda+\mathrm{i}\eta)}{(x-\lambda)^2+\eta^2} \frac{(y-\widetilde{\lambda}\mp\mathrm{i}\eta)}{(y-\widetilde{\lambda})^2+\eta^2} \varrho(\lambda)\widetilde{\varrho}(\widetilde{\lambda})\Phi(\lambda,\widetilde{\lambda})\mathrm{d}\lambda\mathrm{d}\widetilde{\lambda} \\ &=\; \int\!\!\int \frac{(x-\lambda)(y-\widetilde{\lambda}) \pm \eta^2 + \mathrm{i}\eta(y-\widetilde{\lambda}\mp(x-\lambda))}{((x-\lambda)^2+\eta^2)((y-\widetilde{\lambda})^2+\eta^2)} \varrho(\lambda)\widetilde{\varrho}(\widetilde{\lambda})\Phi(\lambda,\widetilde{\lambda})\mathrm{d}\lambda\mathrm{d}\widetilde{\lambda}, \end{aligned} \tag{19}$$

from which we deduce that

$$\mathrm{Re}\big[\psi(x-\mathrm{i}\eta, y+\mathrm{i}\eta) - \psi(x-\mathrm{i}\eta, y-\mathrm{i}\eta)\big] \sim 2\int\!\!\int \frac{\eta\varrho(\lambda)}{(x-\lambda)^2+\eta^2} \frac{\eta\widetilde{\varrho}(\widetilde{\lambda})}{(y-\widetilde{\lambda})^2+\eta^2} \Phi(\lambda,\widetilde{\lambda})\mathrm{d}\lambda\mathrm{d}\widetilde{\lambda}. \tag{20}$$

Finally, the inversion formula follows from Sokhotski-Plemelj identity

$$\lim_{\eta\to 0}\mathrm{Re}\big[\psi(x-\mathrm{i}\eta, y+\mathrm{i}\eta)-\psi(x-\mathrm{i}\eta, y-\mathrm{i}\eta)\big]\sim 2\pi^2\varrho(x)\widetilde{\varrho}(y)\Phi(x,y). \tag{21}$$

Note that the derivation holds for any models of $\mathbf{S}$ and $\widetilde{\mathbf{S}}$ as long as its spectral density converges to a well-defined deterministic limit.

Mean squared overlap for independent sample covariance matrices

Derivation of (4)**.** We now compute the asymptotic expression of the function ψ for sample covariance matrices. Note that we shall omit the identity matrix I_N in our notations throughout the following. We recall that $\mathbf{S}=\sqrt{\mathbf{C}}\boldsymbol{\mathcal{W}}\sqrt{\mathbf{C}}$ and $\widetilde{\mathbf{S}}=\sqrt{\mathbf{C}}\widetilde{\boldsymbol{\mathcal{W}}}\sqrt{\mathbf{C}}$ where $\boldsymbol{\mathcal{W}}$ and $\widetilde{\boldsymbol{\mathcal{W}}}$ are two independent Wishart matrices. Here, we allow these two matrices to be parametrized by a possibly different dimensional ratio q and $\widetilde{q}$. By independence, we have

$$\psi(z,\widetilde{z})=\frac{1}{N}\sum_{k,l}^{N}\big\langle(z-\mathbf{S})^{-1}_{kl}\big\rangle_{\mathcal{P}}\big\langle(\widetilde{z}-\widetilde{\mathbf{S}})^{-1}_{kl}\big\rangle_{\mathcal{P}}, \tag{22}$$

then, we use the *deterministic* estimate like that of [22, 25, 26] to find for any $z=x-\mathrm{i}\eta$ at global scale that

$$\big\langle(z-\mathbf{S})^{-1}_{kl}\big\rangle_{\mathcal{N}}\sim\zeta(z)\Big(z\zeta(z)-\mathbf{C}\Big)^{-1}_{kl}+O(N^{-1/2}), \tag{23}$$

where we recall that $\zeta(z)=1/(1-q+qzg(z))$. Such an estimate holds as well for $\widetilde{\mathbf{S}}$ by replacing q and g with $\widetilde{q}$ and $\widetilde{g}$. Hence, by defining $\widetilde{\zeta}(\widetilde{z})=1/(1-\widetilde{q}+\widetilde{q}\widetilde{z}\widetilde{g}(\widetilde{z}))$, we can rewrite Eq. (22) as

$$\psi(z,\widetilde{z})\sim\frac{1}{N}\operatorname{Tr}[\zeta(z)(z\zeta(z)-\mathbf{C})^{-1}\widetilde{\zeta}(\widetilde{z})(\widetilde{z}\widetilde{\zeta}(\widetilde{z})-\mathbf{C})^{-1}], \tag{24}$$

and then, we use the identity

$$(z\zeta(z)-\mathbf{C})^{-1}(\widetilde{z}\widetilde{\zeta}(\widetilde{z})-\mathbf{C})^{-1}=\frac{1}{\widetilde{z}\widetilde{\zeta}(\widetilde{z})-z\zeta(z)}[(z\zeta(z)-\mathbf{C})^{-1}-(\widetilde{z}\widetilde{\zeta}(\widetilde{z})-\mathbf{C})^{-1}], \tag{25}$$

to obtain

$$\psi(z,\widetilde{z})\sim\frac{\zeta(z)\,\widetilde{\zeta}(\widetilde{z})}{\widetilde{z}\widetilde{\zeta}(\widetilde{z})-z\zeta(z)}\frac{1}{N}\operatorname{Tr}[(z\zeta(z)-\mathbf{C})^{-1}-(\widetilde{z}\widetilde{\zeta}(\widetilde{z})-\mathbf{C})^{-1}]. \tag{26}$$

From this last equation, we deduce

$$\psi(z,\widetilde{z})\sim\frac{1}{\widetilde{z}\widetilde{\zeta}(\widetilde{z})-z\zeta(z)}\left(\widetilde{\zeta}(\widetilde{z})\frac{1}{N}\operatorname{Tr}\Big[\zeta(z)(z\zeta(z)-\mathbf{C})^{-1}\Big]-\zeta(z)\frac{1}{N}\operatorname{Tr}\Big[\widetilde{\zeta}(\widetilde{z}\widetilde{\zeta}(\widetilde{z})-\mathbf{C})^{-1}\Big]\right) \tag{27}$$

By taking the normalized trace in (23), we have at leading order that

$$g(z)\sim\frac{\zeta(z)}{N}\operatorname{Tr}(z\zeta(z)-\mathbf{C})^{-1}\quad\text{and}\quad\widetilde{g}(\widetilde{z})\sim\frac{\widetilde{\zeta}(\widetilde{z})}{N}\operatorname{Tr}(\widetilde{z}\widetilde{\zeta}(\widetilde{z})-\mathbf{C})^{-1}. \tag{28}$$

We therefore conclude by plugging this last equation into (27) that

$$\psi(z,\widetilde{z})\sim\frac{\widetilde{\zeta}(\widetilde{z})g(z)-\zeta(z)\widetilde{g}(\widetilde{z})}{\widetilde{z}\widetilde{\zeta}(\widetilde{z})-z\zeta(z)}, \tag{29}$$

which reduces to Eq. (4) when $q\neq\widetilde{q}$.

Derivation of (6). Using the definition of the complex function m and $\widetilde{m}$ above, we find that $\zeta(z) = 1/(zm(z))$ and $\widetilde{\zeta}(\widetilde{z}) = 1/(\widetilde{z}\widetilde{m}(\widetilde{z}))$. Note that we shall omit the arguments z and $\widetilde{z}$ in m and $\widetilde{m}$ in the following when there are no confusion. First, we rewrite (4) as

$$\psi(z,\widetilde{z}) \sim \frac{1}{1/\widetilde{m} - 1/m}\left[\frac{g}{\widetilde{z}\widetilde{m}} - \frac{\widetilde{g}}{zm}\right], \tag{30}$$

which is equivalent to

$$\psi(z,\widetilde{z}) \sim \frac{1}{z\widetilde{z}}\frac{1}{m-\widetilde{m}}\left[zgm - \widetilde{z}\widetilde{g}\widetilde{m}\right]. \tag{31}$$

Then, we express the function ψ as a function of m and $\widetilde{m}$ so that we have

$$\psi(z,\widetilde{z}) \;\sim\; \frac{1}{q\widetilde{q}z\widetilde{z}}\left[\frac{(\widetilde{q}z - q\widetilde{z})\widetilde{m}^2}{m-\widetilde{m}} + \frac{(q-\widetilde{q})\widetilde{m}}{m-\widetilde{m}}\right] + \frac{m+\widetilde{m}}{q\widetilde{z}} - \frac{1-q}{qz\widetilde{z}}. \tag{32}$$

To use the inversion formula Eq. (3), we need to evaluate $\psi(\lambda - \mathrm{i}\eta, \widetilde{\lambda} \pm \mathrm{i}\eta)$. Using the short handed notation $m_0(\lambda) = \lim_{\eta\to 0} m(\lambda - \mathrm{i}\eta)$, we find

$$\lim_{\eta\to 0}[\psi(\lambda-\mathrm{i}\eta,\widetilde{\lambda}+\mathrm{i}\eta) - \psi(\lambda-\mathrm{i}\eta,\widetilde{\lambda}-\mathrm{i}\eta)]$$
$$\sim \frac{\widetilde{q}\lambda - q\widetilde{\lambda}}{q\widetilde{q}\lambda\widetilde{\lambda}}\frac{m_0\left[\overline{\widetilde{m}_0^2} - \widetilde{m}_0^2\right] + \widetilde{m}_0\overline{\widetilde{m}_0}\left[\overline{\widetilde{m}_0} - \widetilde{m}_0\right]}{\left(m_0 - \overline{\widetilde{m}_0}\right)\left(m_0 - \widetilde{m}_0\right)} + \frac{\widetilde{q}-q}{q\widetilde{q}\lambda\widetilde{\lambda}}\frac{m_0\left[\overline{\widetilde{m}_0} - \widetilde{m}_0\right]}{\left(m_0 - \overline{\widetilde{m}_0}\right)\left(m_0 - \widetilde{m}_0\right)} + \mathrm{Im}, \tag{33}$$

s where we omit the explicit expressions of the imaginary part since this is useless (see Eq. (3)). Then, using the representation $m_0 = m_R + \mathrm{i}m_I$ and $\widetilde{m}_0 = \widetilde{m}_R + \mathrm{i}\widetilde{m}_I$, one finds

$$\begin{aligned} m_0\left[\overline{\widetilde{m}_0^2} - \widetilde{m}_0^2\right] + \widetilde{m}_0\overline{\widetilde{m}_0}\left[\overline{\widetilde{m}_0} - \widetilde{m}_0\right] &= 2\widetilde{m}_I\left[2m_I\widetilde{m}_R + \mathrm{i}(\widetilde{m}_R^2 + \widetilde{m}_R^2 - 2m_R\widetilde{m}_R)\right], \\ \overline{m_0}\left[\widetilde{m}_0 - \overline{\widetilde{m}_0}\right] &= 2\widetilde{m}_I\left[m_I - \mathrm{i}m_R\right], \end{aligned} \tag{34}$$

and

$$\left(m_0 - \overline{\widetilde{m}_0}\right)\left(m_0 - \widetilde{m}_0\right) = (m_R - \widetilde{m}_R)^2 - (m_I^2 - \widetilde{m}_I^2) + 2\mathrm{i}m_I(m_R - \widetilde{m}_R). \tag{35}$$

Straightforward computations yields

$$\begin{aligned} \left|\left(m_0 - \overline{\widetilde{m}_0}\right)\left(m_0 - \widetilde{m}_0\right)\right|^2 &= \left[(m_R - \widetilde{m}_R)^2 - (m_I^2 - \widetilde{m}_I^2)\right]^2 + 4m_I^2(m_R - \widetilde{m}_R)^2, \\ &= \left[(m_R - \widetilde{m}_R)^2 + (m_I^2 + \widetilde{m}_I^2)^2\right]\left[(m_R - \widetilde{m}_R)^2 + (m_I^2 - \widetilde{m}_I^2)^2\right], \end{aligned} \tag{36}$$

which is exactly the denominator in (6). For the numerator, elementary complex analysis in Eq. (33) yields

$$\left(m_0\left[\overline{\widetilde{m}_0^2} - \widetilde{m}_0^2\right] + \widetilde{m}_0\overline{\widetilde{m}_0}\left[\overline{\widetilde{m}_0} - \widetilde{m}_0\right]\right) \times \overline{\left(m_0 - \overline{\widetilde{m}_0}\right)\left(m_0 - \widetilde{m}_0\right)} = 4m_I\widetilde{m}_I\left[m_R|\widetilde{m}_0|^2 - \widetilde{m}_R|m_0|^2\right], \tag{37}$$

and

$$m_0\left[\overline{\widetilde{m}_0} - \widetilde{m}_0\right] \times \overline{\left(m_0 - \overline{\widetilde{m}_0}\right)\left(m_0 - \widetilde{m}_0\right)} = 2m_I\widetilde{m}_I\left[|\widetilde{m}_0|^2 - |m_0|^2\right]. \tag{38}$$

By regrouping these last three equations with the prefactors in (33), and recalling that $m_I(\lambda) = \pi q\varrho(\lambda)$ and $\widetilde{m}_I(\widetilde{\lambda}) = \pi\widetilde{q}\widetilde{\varrho}(\widetilde{\lambda})$, so we obtain by using the inversion formula (3) the general result:

$$\Phi_{q,\widetilde{q}}(\lambda,\widetilde{\lambda}) = \frac{2(\widetilde{q}\lambda - q\widetilde{\lambda})\left[m_R|\widetilde{m}_0|^2 - \widetilde{m}_R|m_0|^2\right] + (\widetilde{q}-q)\left[|\widetilde{m}_0|^2 - |m_0|^2\right]}{\lambda\widetilde{\lambda}\left[(m_R - \widetilde{m}_R)^2 + (m_I + \widetilde{m}_I)^2\right]\left[(m_R - \widetilde{m}_R)^2 + (m_I - \widetilde{m}_I)^2\right]}. \tag{39}$$

One easily retrieve Eq. (6) by taking $\widetilde{q} = q$. Another interesting case if when $\widetilde{q} = 0$ as we expect $\widetilde{\lambda} \to \mu$, which is precisely the framework of [15]. In that case, we have $\widetilde{m}_R = 1/\mu$ and $\widetilde{m}_I = 0$. Hence, we deduce from (39) that

$$\begin{aligned}\Phi_{q,\widetilde{q}=0}(\lambda,\mu) &= \frac{q}{\lambda\mu\big[(m_R - 1/\mu)^2 + m_I^2\big]} \\ &= \frac{q\mu}{\lambda|1-\mu m_0(\lambda)|^2},\end{aligned} \tag{40}$$

which is exactly the result of [15].

Derivation of (9). We set $\widetilde{q} = q$ and $\widetilde{\lambda} = \lambda + \varepsilon$ with $\varepsilon > 0$. This yields

$$\begin{aligned}m_R(\widetilde{\lambda}) &= m_R(\lambda) + \varepsilon\partial_\lambda m_R(\lambda) + O(\varepsilon^2), \\ |m_0(\widetilde{\lambda})|^2 &= |m_0(\lambda)|^2 + 2\varepsilon\big[m_R(\lambda)\partial_\lambda m_R(\lambda) + \pi^2 q^2 \varrho(\lambda)\partial_\lambda\varrho(\lambda)\big] + O(\varepsilon^2).\end{aligned} \tag{41}$$

Thus, we get from (7) for $q = \widetilde{q}$ that

$$\begin{aligned}\alpha(\lambda,\lambda+\varepsilon) &= -\varepsilon^2\Big[2m_R(\lambda)[m_R(\lambda)\partial_\lambda m_R'(\lambda) + m_I(\lambda)\partial_\lambda m_I'(\lambda)] - \partial_\lambda m_R(\lambda)(m_R^2(\lambda) + m_I^2(\lambda))\Big] + O(\varepsilon^3) \\ &= |m_0|^2\partial_\lambda m_R - m_R\partial_\lambda|m_0|^2 + O(\varepsilon^3),\end{aligned} \tag{42}$$

and

$$\begin{aligned}\gamma(\lambda,\lambda+\varepsilon) &= \Big[\varepsilon^4\partial_\lambda m_R(\lambda)^2 + 4\,\varepsilon^2\, m_I^2\big((\partial_\lambda a(\lambda))^2 + (\partial_\lambda m_I)^2\big)\Big] + O(\varepsilon^3) \\ &= 4\varepsilon^2 m_I^2(\lambda)|\partial_\lambda m_0(\lambda)|^2 + O(\varepsilon^3).\end{aligned} \tag{43}$$

The result (9) follows by plugging Eqs. (42) and (43) into Eq. (6) and then set $\varepsilon = 0$.

Mean squared overlap for deformed GOE matrices

The derivation of the overlaps (8) for two independent deformed GOEs is very similar to sample covariance matrices. Hence, we shall omit most details that can be obtained by following the arguments of the above Appendix. Again, we shall skip the arguments λ and $\widetilde{\lambda}$ where there is no confusion.

For completeness, we recall some notations. We defined g and $\widetilde{g}$ are respectively the Stieltjes transform of $\mathbf{S} = \mathbf{C} + \mathbf{W}$ and $\widetilde{\mathbf{S}} = \mathbf{C} + \widetilde{\mathbf{W}}$ where the noises are independent. Also, we introduced

$$\xi(z) = z - \sigma^2 g(z), \qquad \widetilde{\xi}(\widetilde{z}) = \widetilde{z} - \widetilde{\sigma}^2 g(\widetilde{z}), \tag{44}$$

where σ^2 can be different from $\widetilde{\sigma}^2$. We still use the convention $\xi_0(\lambda) = \lim_{\eta\downarrow 0}\xi(\lambda - \mathrm{i}\eta) \equiv \xi_R + \mathrm{i}\xi_I$ and $\widetilde{\xi}_0(\lambda) = \lim_{\eta\downarrow 0}\xi(\widetilde{\lambda} - \mathrm{i}\eta) \equiv \widetilde{\xi}_R + \mathrm{i}\widetilde{\xi}_I$. It is easy to see that $\xi_R = \lambda - \sigma^2 g_R$ and $\xi_I = -\sigma^2 g_I$.

For deformed GOE, the resolvent can also be approximated by a deterministic matrix for $N \to \infty$. Indeed, one has [22, 26]:

$$\big\langle (z - \mathbf{S})^{-1}_{kl}\big\rangle_{\mathcal{P}} \sim \Big(\xi(z) - \mathbf{C}\Big)^{-1}_{kl} \tag{45}$$

and $\big\langle(\widetilde{z} - \widetilde{\mathbf{S}})^{-1}_{kl}\big\rangle_{\mathcal{P}}$ is obtained from Eq. (45) by replacing ξ by $\widetilde{\xi}$. Then, plugging this into (2), we obtain,

$$\begin{aligned}\lim_{\eta\to 0}\big[\psi(\lambda - \mathrm{i}\eta, \widetilde{\lambda} + \mathrm{i}\eta) - \psi(\lambda - \mathrm{i}\eta, \widetilde{\lambda} - \mathrm{i}\eta)\big] &= \frac{g_0 - \overline{\widetilde{g}_0}}{\overline{\widetilde{\xi}} - \xi} - \frac{g_0 - \widetilde{g}_0}{\widetilde{\xi} - \xi} \\ &= \frac{g_0\Big(\widetilde{\xi} - \overline{\widetilde{\xi}}\Big) + \xi\Big(\overline{\widetilde{g}_0} - \widetilde{g}_0\Big) + g_0\,\overline{\widetilde{\xi}} - \overline{\widetilde{g}_0}\,\widetilde{\xi}}{\Big(\overline{\widetilde{\xi}} - \xi\Big)\Big(\widetilde{\xi} - \xi\Big)}\end{aligned} \tag{46}$$

We proceed as above (see Eq. (33) and thereafter) to find

$$g_0\left(\widetilde{\xi}-\overline{\widetilde{\xi}}\right)+\xi\left(\overline{\widetilde{g}_0}-\widetilde{g}_0\right)+g_0\,\overline{\widetilde{\xi}}-\overline{\widetilde{g}_0}\,\widetilde{\xi}=2\big[(\xi_I\widetilde{g}_I-g_I\widetilde{\xi}_I)+\mathrm{i}\big(\widetilde{\xi}_I(g_R-\widetilde{g}_R)-\widetilde{g}_I(\xi_R-\widetilde{\xi}_R)\big)\big] \tag{47}$$

and

$$\left(\overline{\widetilde{\xi}}-\xi\right)\left(\widetilde{\xi}-\xi\right)=(\xi_R-\widetilde{\xi}_R)^2+(\widetilde{\xi}_I^2-\xi_I^2)+2\mathrm{i}\xi_I(\widetilde{\xi}_R-\xi_R). \tag{48}$$

Hence, by putting these last two equations into (46) and then using (3), we get after some straightforward computations

$$\Phi_a(\lambda,\widetilde{\lambda})=\frac{(\sigma^2+\widetilde{\sigma}^2)(\xi_R-\widetilde{\xi}_R)^2+2\sigma^2\widetilde{\sigma}^2(g_R-\widetilde{g}_R)(\xi_R-\widetilde{\xi}_R)-(\sigma^2-\widetilde{\sigma}^2)(\xi_I^2-\widetilde{\xi}_I^2)}{\big[(\xi_R-\widetilde{\xi}_R)^2+(\xi_I+\widetilde{\xi}_I)^2\big]\big[(\xi_R-\widetilde{\xi}_R)^2+(\xi_I-\widetilde{\xi}_I)^2\big]}, \tag{49}$$

which is the generalization of Eq. (8) for two different noise parameters $\sigma\neq\widetilde{\sigma}$. If we now consider $\sigma=\widetilde{\sigma}$, we have

$$\begin{aligned}\Phi_a(\lambda,\widetilde{\lambda}) &= 2\sigma^2\frac{(\xi_R-\widetilde{\xi}_R)^2+\sigma^2(g_R-\widetilde{g}_R)(\xi_R-\widetilde{\xi}_R)}{\big[(\xi_R(\lambda)-\xi_R(\widetilde{\lambda}))^2+(\xi_I(\lambda)+\xi_I(\widetilde{\lambda}))^2\big]\big[(\xi_R(\lambda)-\xi_R(\widetilde{\lambda}))^2+(\xi_I(\lambda)-\xi_I(\widetilde{\lambda}))^2\big]},\\ &= \frac{2\sigma^2(\lambda-\widetilde{\lambda})(\xi_R(\lambda)-\xi(\widetilde{\lambda}))}{\big[(\xi_R(\lambda)-\xi_R(\widetilde{\lambda}))^2+(\xi_I(\lambda)+\xi_I(\widetilde{\lambda}))^2\big]\big[(\xi_R(\lambda)-\xi_R(\widetilde{\lambda}))^2+(\xi_I(\lambda)-\xi_I(\widetilde{\lambda}))^2\big]},\end{aligned} \tag{50}$$

that is exactly Eq. (8). As for sample covariance matrices, one can again set $\widetilde{\lambda}=\lambda+\varepsilon$ to find

$$\Phi_a(\lambda,\lambda)=\frac{\sigma^2\partial_\lambda\xi_R(\lambda)}{2\xi_I^2\big([\partial_\lambda\xi_R(\lambda)]^2+[\partial_\lambda\xi_I(\lambda)]^2\big)}, \tag{51}$$

and this is exactly (10). Now, if we consider $\widetilde{\sigma}^2=0$ (no noise), it implies that $\widetilde{m}_I=0$ and $\widetilde{m}_R=\mu$. Then, one can easily check that this yields in Eq. (49):

$$\Phi_a(\lambda,\mu)=\frac{\sigma^2}{(\xi_R-\mu)^2+\xi_I^2}, \tag{52}$$

which is exactly the result derived in [18, 23, 24].

The case of correlated Gaussian additive noises

We define

$$\mathbf{S}=\mathbf{C}+\mathbf{W}_1,\quad \widetilde{\mathbf{S}}=\mathbf{C}+\mathbf{W}_2, \tag{53}$$

where $\mathbf{W}_1,\mathbf{W}_2$ are two correlated GOE matrices (independent from $\mathbf{C}$) satisfying

$$\langle\mathbf{W}_1\rangle_{\mathcal{N}}=0,\quad \langle\mathbf{W}_2\rangle_{\mathcal{N}}=0,\quad \mathrm{Cov}(\mathbf{W}_1,\mathbf{W}_2)=\begin{pmatrix}\sigma_1^2 & \rho_{12}\\ \rho_{12} & \sigma_2^2\end{pmatrix}, \tag{54}$$

where we denoted by $\mathcal{N}$ the Gaussian measure and used the abbreviation $\rho_{12}=\rho\sigma_1\sigma_2$. Using the stability of GOE under addition, let us rewrite the noise terms as

$$\begin{aligned}\mathbf{W}_1 &= \mathbf{A}+\mathbf{B}_1,\\ \mathbf{W}_2 &= \mathbf{A}+\mathbf{B}_2,\end{aligned} \tag{55}$$

where $\mathbf{A}$ that satisfies

$$\langle\mathbf{A}\rangle_{\mathcal{N}}=0,\quad \langle\mathbf{A}^2\rangle_{\mathcal{N}}=\rho_{12}, \tag{56}$$

and $\mathbf{B}_1$ and $\mathbf{B}_2$ are two GOEs matrices independent from $\mathbf{A}$ with

$$\begin{cases}\langle\mathbf{B}_1\rangle_{\mathcal{N}}=0,\quad \langle\mathbf{B}_2\rangle_{\mathcal{N}}=0,\\ \langle\mathbf{B}_1^2\rangle_{\mathcal{N}}=\sigma_1^2-\rho_{12},\quad \langle\mathbf{B}_2^2\rangle_{\mathcal{N}}=\sigma_2^2-\rho_{12},\\ \langle\mathbf{B}_1\mathbf{B}_2\rangle_{\mathcal{N}}=0.\end{cases} \tag{57}$$

One can check that this parametrization yields exactly the correlation structure of Eq. (54). Therefore, using (55) into (53), we have the equivalence (in law)

$$\mathbf{S}_1 = \mathbf{D} + \mathbf{B}_1, \quad \widetilde{\mathbf{S}} = \mathbf{D} + \mathbf{B}_2, \qquad \mathbf{D} := \mathbf{C} + \mathbf{A}. \tag{58}$$

Since the noises are now independent and that the mean squared overlap Φ_a, given in Eq. (49), is "independent" from the exact structure of $\mathbf{C}$, we can therefore replace $\mathbf{C}$ by $\mathbf{D}$. Hence, we deduce that the overlaps for this model will again be given by Eq. (49) with $\sigma^2 = \sigma_1^2 - \rho_{12}$, and $\widetilde{\sigma}^2 = \sigma_2^2 - \rho_{12}$. If $\sigma = \widetilde{\sigma}$, then, Φ_a is given by Eq. (8) with variance $\sigma^2(1-\rho)$, as announced.